

# THE CAUCHY PROBLEM FOR WAVE MAPS ON HYPERBOLIC SPACE IN DIMENSIONS $d \geq 4$

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**ABSTRACT.** We establish global well-posedness and scattering for wave maps from  $d$ -dimensional hyperbolic space into Riemannian manifolds of bounded geometry for initial data that is small in the critical Sobolev space for  $d \geq 4$ . The main theorem is proved using the moving frame approach introduced by Shatah and Struwe. However, rather than imposing the Coulomb gauge we formulate the wave maps problem in Tao's caloric gauge, which is constructed using the harmonic map heat flow. In this setting the caloric gauge has the remarkable property that the main 'gauged' dynamic equations reduce to a system of nonlinear *scalar* wave equations on  $\mathbb{H}^d$  that are amenable to Strichartz estimates rather than *tensorial* wave equations (which arise in other gauges such as the Coulomb gauge) for which useful dispersive estimates are not known. This last point makes the heat flow approach crucial in the context of wave maps on curved domains.

## 1. INTRODUCTION

Let  $(\mathcal{M}, \eta)$  be a Lorentzian manifold with metric  $\eta$  given by the product  $\mathcal{M} = \mathbb{R} \times \Sigma$  where  $\Sigma = (\Sigma, \mathbf{h})$  is a Riemannian manifold of dimension  $d$  with metric  $\mathbf{h}$ . Let  $(\mathcal{N}, \mathbf{g})$  be a complete Riemannian manifold without boundary of dimension  $n$ . A wave map  $u : \mathcal{M} \rightarrow \mathcal{N}$  is a formal critical point of the Lagrangian action

$$\mathcal{L}(u) := \int_{\mathcal{M}} \langle du, du \rangle_{\eta^{-1} \otimes u^* \mathbf{g}} \, \mathrm{dvol}_{\eta}. \quad (1.1)$$

Above we view the differential  $du$  as a section of the vector bundle  $T^* \mathcal{M} \otimes u^* T\mathcal{N}$ , which is endowed with the metric  $\eta^{-1} \otimes u^* \mathbf{g}$ . Here  $u^* T\mathcal{N}$  denotes the pullback of  $T\mathcal{N}$  by the map  $u$  and  $u^* \mathbf{g}$  is the pullback metric. In coordinates  $\{x^\alpha\}_{\alpha=0}^d$  on  $\mathcal{M}$  and  $\{u^j\}_{j=1}^n$  on  $\mathcal{N}$  the Lagrangian density becomes

$$\langle du, du \rangle_{\eta^{-1} \otimes u^* \mathbf{g}} \, \mathrm{dvol}_{\eta} = \eta^{\alpha\beta} \mathbf{g}_{jk}(u) \partial_\alpha u^j \partial_\beta u^k \sqrt{|\eta|} \, dx \quad (1.2)$$

The Euler-Lagrange equations for (1.1) are given by

$$\eta^{\alpha\beta} D_\alpha \partial_\beta u = 0 \quad (1.3)$$

where  $D$  is the pullback covariant derivative on  $u^* T\mathcal{N} \otimes T^* \mathcal{M}$ . Expanding the above we obtain following system of nonlinear wave equations on  $\mathcal{M} = \mathbb{R} \times \Sigma$ :

$$\square_{\Sigma} u^k = \eta^{\alpha\beta} (\mathcal{N}) \Gamma_{ij}^k(u) \partial_\alpha u^i \partial_\beta u^j$$

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where  $\square_\Sigma = -\partial_t^2 + \Delta_\Sigma$  is the D'Alembertian on  $\mathcal{M}$ ,  $\Delta_\Sigma$  is the Laplace-Beltrami operator on  $\Sigma$ , and the  ${}^{(\mathcal{N})}\Gamma_{ij}^k$  are the Christoffel symbols associated to the metric connection on  $T\mathcal{N}$ .

Equivalently, by the Nash embedding theorem we can view  $\mathcal{N}$  as an isometrically embedded submanifold of  $(\mathbb{R}^N, \langle \cdot, \cdot \rangle)$ . In this case we write  $u = (u^1, \dots, u^N)$  where  $u : \mathcal{M} \rightarrow \mathcal{N} \hookrightarrow \mathbb{R}^N$ . A wave map  $u$  is then a solution to

$$\square_\Sigma u \perp T_u \mathcal{N} \quad (1.4)$$

which can be expressed in coordinates as

$$\square_\Sigma u^K = \eta^{\alpha\beta} S_{IJ}^K(u) \partial_\alpha u^I \partial_\beta u^J \quad (1.5)$$

where  $S = S_{IJ}^K$  is the second fundamental form of the embedding  $\mathcal{N} \hookrightarrow \mathbb{R}^N$ .

In this paper we consider the domain

$$\Sigma = \mathbb{H}^d, \quad d \geq 4$$

where  $\mathbb{H}^d$  is  $d$ -dimensional hyperbolic space. To simplify the exposition we restrict to dimension  $d = 4$  and  $\Sigma = \mathbb{H}^4$ , noting that the results and methods used here apply equally well to domains  $\Sigma = \mathbb{H}^d$  for all  $d \geq 4$ . To avoid technical distractions, we also assume that  $\mathcal{N}$  is a *closed* manifold (i.e., compact without boundary), which ensures uniform boundedness of the second fundamental form  $S(u)$  and its derivatives in the Nash embedding. We refer to Remark 3 for a discussion on the case when  $\mathcal{N}$  is non-compact.

We study the Cauchy problem for (1.4) for smooth finite energy initial data  $(u, \partial_t u)|_{t=0} = (u_0, u_1)$  given by maps

$$u_0 : \mathbb{H}^4 \rightarrow \mathcal{N}, \quad u_1 : \mathbb{H}^4 \rightarrow u_0^* T\mathcal{N}, \quad u_1(x) \in T_{u_0(x)} \mathcal{N} \quad (1.6)$$

This means that we can think of the pair of initial data  $(u_0, u_1)$  as a map into the tangent space of the target  $T\mathcal{N}$  and we will also write  $\vec{u}(0) := (u_0, u_1) : \mathbb{H}^4 \rightarrow T\mathcal{N}$ . We will further assume that we can find a point  $u_\infty := u_0(\infty) \in \mathcal{N}$  and a compact subset  $K \Subset \mathbb{H}^4$  so that

$$(u_0(x), u_1(x)) = (u_0(\infty), 0) \in T_{u_0(\infty)} \mathcal{N} \quad \text{if } x \in \mathbb{H}^4 - K, \quad (1.7)$$

that is, the data is a constant map in the exterior of a compact subset of  $\mathbb{H}^4$ .

A simple version of our main result is as follows.

**Theorem 1.1.** *There exists an  $\epsilon_0 > 0$  small enough so that for all smooth  $(u_0, u_1)$  as in (1.6), (1.7) with*

$$\|(du_0, u_1)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} < \epsilon_0$$

*there exists a unique, global smooth solution  $u : \mathbb{R} \times \mathbb{H}^4 \rightarrow \mathcal{N} \subset \mathbb{R}^N$  to (1.4) with initial data  $(u, \partial_t u)|_{t=0} = (u_0, u_1)$  and satisfying*

$$\sup_{t \in \mathbb{R}} \|(du(t), \partial_t u(t))\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \lesssim \epsilon_0.$$

*Moreover we have the following qualitative pointwise decay*

$$\|u(t, \cdot) - u_0(\infty)\|_{L_x^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty.$$

Theorem 1.1 is a global well-posedness and scattering result for small critical Sobolev data, which is of similar flavor to the results of [31, 51, 58, 68] in the flat high dimensional case ( $\Sigma = \mathbb{R}^d$  with  $d \geq 4$ ). A major motivation for the latter results was to develop techniques to approach the two dimensional case  $\Sigma = \mathbb{R}^2$ , which is challenging due to weaker dispersion but particularly interesting in view of energy criticality of the wave maps equation. In fact, based on the small critical Sobolev data results [36, 69, 74], a satisfactory large data result on  $\mathbb{R} \times \mathbb{R}^2$  was recently established in [37, 64, 65, 71]. By the same token, we view Theorem 1.1 as a first step towards the interesting case  $\Sigma = \mathbb{H}^2$ , whose study was initiated by the authors [41–43] under symmetry assumptions. We refer to Section 1.2 for a more detailed discussion of the history and motivation of the problem.

A particularly simple proof of the flat space version of Theorem 1.1 was given by Shatah and Struwe [58], which relied on the use of the derivative formulation of the wave maps equation in the Coulomb gauge (see also [51] for a similar approach). However the analysis of [58] does not extend to our case, due to tensoriality of the main equations. Instead, we formulate the wave map problem in Tao’s caloric gauge, which leads to equations with scalar principal part with nonlinearity amenable to perturbative analysis. In view of the effectiveness of the caloric gauge on  $\mathbb{R} \times \mathbb{R}^2$  [71], we expect the framework developed in this paper to be useful in the important case of  $\Sigma = \mathbb{H}^2$  as well. In Section 1.1 below, we will give a further explanation of the main difficulties of our problem, and how Tao’s caloric gauge provides their resolution.

*Remark 1.* In Section 4 we will give a more precise formulation of Theorem 1.1 that includes the boundedness of appropriate Strichartz-type norms in the caloric gauge as well as a notion of scattering. Note that the norm  $H^1(\mathbb{H}^4; T\mathcal{N})$  is a Sobolev space, which will be defined in (2.41) in Section 2.5.

*Remark 2.* Theorem 1.1 can be easily extended to wave maps  $u : \mathbb{R} \times \mathbb{H}^d \rightarrow \mathcal{N}$  for all  $d \geq 4$  using arguments identical to those contained in this paper. We have chosen to present the proof only for dimension  $d = 4$  to keep the exposition as simple as possible. Unfortunately, the techniques presented here do not suffice to obtain the same results in dimensions  $d = 2, 3$ , due to the failure of favorable Strichartz estimates (see Theorem 4.2).

*Remark 3.* The restriction to a compact target manifold  $\mathcal{N}$  is not expected to be essential, and Theorem 1.1 should hold for any smooth Riemannian manifold  $\mathcal{N}$  with bounded geometry (i.e., the curvature tensor and its covariant derivatives are uniformly bounded). One way to extend our proof to the noncompact case would be to replace the a priori estimates we proved using the isometric embedding by their intrinsic counterparts, as was done in the case of the flat domain  $\Sigma = \mathbb{R}^d$  [61, 71]. To avoid distracting technicalities, we will not address this issue in detail.

**1.1. Main ideas.** A key difficulty for establishing well-posedness of the wave maps equation at the critical Sobolev regularity is the presence of a quadratic nonlinearity with derivatives (henceforth referred to as a ‘nonperturbative part’ of the nonlinearity) which precludes a direct iteration argument based on estimates for  $\square_\Sigma$ ; see [17].

In the breakthrough work of Tao [68, 69], it was realized that the gauge structure of the wave maps equation can be used to renormalize away the nonperturbative part of the nonlinearity. This idea has its roots in the harmonic maps literature,

e.g., see Hélein [23]. The work of Shatah and Struwe [58] and Nahmod, Stefanov and Uhlenbeck [51] further clarified the picture by exhibiting a particular representation of the wave maps equation, namely the *derivative formulation* in the *Coulomb gauge*, in which the whole nonlinearity is essentially cubic or higher. It was shown in [51, 58] that in this formulation, the wave maps equation is amenable to perturbative analysis based on  $\square_\Sigma$  in the small critical Sobolev data setting on  $\mathbb{R} \times \mathbb{R}^d$  when  $d \geq 4$ . We begin by summarizing the main ideas of this approach, as it will provide a starting point for our discussion.

*Derivative formulation and the Coulomb gauge.* In the derivative formulation, one works with the differential  $du$  instead of the map  $u$  itself. Being a section of the vector bundle  $u^*TN$  (pullback of the tangent bundle  $TN$  by  $u$ ), the differential  $du$  has the advantage of being linear. With respect to a global field  $e$  of orthonormal frames of  $u^*TN$ , which exists since the domain  $\mathbb{R} \times \Sigma$  is contractible,  $du$  can be conveniently written as an  $\mathbb{R}^n$ -valued 1-form  $\Psi = \psi_\alpha dx^\alpha$ . Denoting by  $\mathbf{D}_\alpha$  the pullback covariant derivative of  $u^*TN$  in the frame  $e$ , the wave maps equation can be rephrased as the div-curl system<sup>1</sup>

$$\eta^{\alpha\beta} \mathbf{D}_\alpha \psi_\beta = 0, \quad \mathbf{D}_\alpha \psi_\beta - \mathbf{D}_\beta \psi_\alpha = 0 \quad (1.8)$$

The nonlinearity in this formulation is encoded in  $\mathbf{D}_\alpha$ , which takes the form  $\partial_\alpha + A_\alpha$  for some real skew-symmetric  $n \times n$ -matrix valued 1-form  $A_\alpha dx^\alpha$  (connection 1-form). Considering the action of the commutator  $[\mathbf{D}_\alpha, \mathbf{D}_\beta]$  on sections of  $u^*TN$ , it follows that  $A_\alpha$  obeys the curl equation

$$\nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta] = \mathbf{R}(u)(\psi_\alpha, \psi_\beta) \quad (1.9)$$

where  $\mathbf{R}(u)$  is the pullback Riemann curvature. However,  $A_\alpha$  is otherwise undetermined. The gauge invariance of the wave maps equation manifests as the freedom of the choice of  $e$ , which in turn determines  $A_\alpha$ .

The Coulomb gauge is given by a frame  $e$  with a connection 1-form  $A$  whose restriction to the constant- $t$  hypersurfaces  $\{t\} \times \Sigma$  has zero divergence, i.e.,

$$\mathbf{h}^{ab} \nabla_a A_b = 0 \quad (1.10)$$

Such a frame always exists in the small critical data setting by the classical argument of Uhlenbeck [75]. By the div-curl system (1.9), (1.10), it follows that  $A$  can be recovered from  $\psi_\alpha$  by solving the nonlinear elliptic equations

$$\mathbf{h}^{bc} \nabla_b \nabla_c A_\alpha - \mathbf{h}^{bc} [\nabla_b, \nabla_c] A_\alpha = \mathbf{h}^{bc} \nabla_b \left( \mathbf{R}(u)(\psi_c, \psi_\alpha) - [A_c, A_\alpha] \right) \quad (1.11)$$

On the other hand, from the div-curl system (1.8), we may derive the nonlinear wave equation

$$\begin{aligned} \eta^{\beta\gamma} \nabla_\beta \nabla_\gamma \psi_\alpha - \eta^{\beta\gamma} [\nabla_\beta, \nabla_\gamma] \psi_\alpha &= -2\eta^{\beta\gamma} A_\beta \nabla_\gamma \psi_\alpha - (\eta^{\beta\gamma} \nabla_\beta A_\gamma) \psi_\alpha \\ &\quad - \eta^{\beta\gamma} A_\beta A_\gamma \psi_\alpha + \eta^{\beta\gamma} \mathbf{R}(u)(\psi_\beta, \psi_\alpha) \psi_\gamma \end{aligned} \quad (1.12)$$

In the flat case  $\Sigma = \mathbb{R}^d$ , the left-hand sides of (1.11) and (1.12) reduce to well-understood linear operators, namely the (componentwise) scalar Laplacian and d'Alembertian, respectively. As a consequence, when  $d \geq 4$  the coupled system

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<sup>1</sup>We adopt the usual index notation for tensors, where the greek letters  $\alpha, \beta, \dots$  correspond to the global space-time coordinates  $x^\alpha = (t = x^0, x^1, \dots, x^d)$  and the latin letters  $a, b, \dots$  represent the space coordinates  $x^a = (x^1, \dots, x^d)$ . For further explanation of the notation, we refer to Section 2.

(1.11) and (1.12) can be used to establish appropriate critical regularity a priori bounds for the smooth wave map  $u$  with the given smooth initial data, via a direct perturbative analysis based on  $L^p$  estimates for  $A_\alpha, \nabla A_\alpha$  and global Strichartz estimates for  $\psi_\alpha$ <sup>2</sup>. Global existence and scattering of  $u$  can then be inferred from these a priori bounds, which completes the proof of Theorem 1.1 in the case  $\Sigma = \mathbb{R}^d$  and  $d \geq 4$ . We refer to [51, 58] for more details.

*Issue of tensoriality on a curved background.* This type of analysis, however, does not immediately extend to curved  $\Sigma$  in general. The main issue is *tensoriality* of  $\psi_\alpha$  and  $A_\alpha$ , which is inconspicuous in the flat case but becomes pronounced on a curved background. First of all, observe that the left-hand side of (1.12), which is the Hodge-de Rham d'Alembertian on 1-forms, does not coincide with the (componentwise) scalar d'Alembertian on  $\mathbb{R} \times \Sigma$  in general. In particular, unlike the scalar d'Alembertian, no useful dispersive estimates, such as *global* Strichartz estimates, seem to be known for this linear operator in our setting  $\Sigma = \mathbb{H}^d$ . To make matters worse, the left-hand side of (1.11), which is the Hodge-de Rham Laplacian on 1-forms, is also distinct from the scalar Laplacian in general. Even in the special case  $\Sigma = \mathbb{H}^d$ , much less is known about  $L^p$  estimates for the Hodge-de Rham Laplacian compared to the scalar Laplacian; e.g., see [11].

Nevertheless, there is a reason to believe that this obstruction is not essential, on the ground that the wave map itself is *not* tensorial, i.e., its transformation laws under coordinate changes of the domain are those of a scalar. Indeed, we will show in this paper that a geometrically natural realization of  $u$  as a linear *scalar* field with (in fact, more) favorable nonlinearity can be obtained by a different choice of gauge, namely the *caloric gauge* introduced by Tao [70, 71].

*Idea of the caloric gauge.* The main component, as well as the namesake, of the caloric gauge is the *harmonic map heat flow*, a celebrated geometric PDE in its own right. In simplest terms, the idea of the caloric gauge can be phrased as follows. Given a smooth wave map  $u(t, x)$  defined on a time interval  $I$ , we introduce a new time parameter  $s \in \mathbb{R}^+$  and solve the harmonic map heat flow in  $s$  with initial data  $u(t, x)$ . Since the initial data  $u(t, x)$  is suitably small, the heat flow can be shown to converge to a fixed point  $u_\infty$  on  $\mathcal{N}$  as  $s \rightarrow \infty$ . This now gives a canonical choice of frame at  $s = \infty$ , namely the *same set of orthonormal vectors*  $e(\infty, t, x) = e_\infty$  for each point  $(t, x) \in I \times \Sigma$ . The *caloric gauge* determined by the choice  $e_\infty$  is then constructed by parallel transporting the frame  $e_\infty = e(\infty, t, x)$  from  $s = \infty$  back along the harmonic map flow.

The reason why the caloric gauge is so effective can be understood clearly by making an analogy between the harmonic map heat flow resolution and the Littlewood-Paley theory. One way to define Littlewood-Paley theory on a general complete Riemannian manifold  $\Sigma$  is to introduce a new heat-time parameter  $s \in \mathbb{R}^+$  and to use the linear heat equation  $(\partial_s - \Delta_\Sigma)f = 0$ ; see, for instance, [32, 63]. The main ideas can be sketched as follows. Given a function  $g$  on  $\Sigma$  and the solution  $f$  to the linear heat equation with  $f|_{s=0} = g$ , we interpret  $f(s)$  as the projection of  $g$  to frequencies less than  $s^{-1/2}$ , which may be justified in the case  $\Sigma = \mathbb{R}^d$

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<sup>2</sup>We remark that a similar scheme also works in dimensions  $d = 2, 3$ , but the proof is far more complex as global Strichartz estimates are insufficient; it is now necessary to reveal the null structure of the wave maps equation in the Coulomb gauge, and utilize multilinear null form estimates. See [35, 36] for details.

by observing that the Fourier transform of  $f(s)$  is simply that of  $g$  multiplied by a rescaled Gaussian adapted to the ball  $\{|\xi| \simeq s^{-1/2}\}$ . A Littlewood-Paley projection to frequencies comparable to  $s^{-1/2}$  is defined to be  $f^{(k)}(s) := s^k \partial_s^k f(s) = s^k \Delta^k f(s)$  for a fixed  $k \geq 1$ , the idea being that  $s^k \Delta^k$  damps the frequencies much lower than  $s^{-1/2}$ . Using the fact that  $f \rightarrow 0$  as  $s \rightarrow \infty$  and performing Taylor expansion at  $s = \infty$ , one obtains the reconstruction formula  $g = \int_0^\infty \frac{(-1)^k}{(k-1)!} f^{(k)}(s) \frac{ds}{s}$ . It can be readily seen that the main results of Littlewood-Paley theory (e.g., the square function theorem) extends to this heat flow definition.

In analogy with this formulation of Littlewood-Paley theory, we may interpret the harmonic map heat flow resolution of a map  $u$  as a geometric Littlewood-Paley decomposition of  $u$ . The analogue of the Littlewood-Paley projection  $f^{(1)}(s)$  is  $s \partial_s u(s)$ , which is a section of  $u(s)^* T\mathcal{N}$ . Given a frame  $e(s)$  of this bundle,  $s \partial_s u(s)$  is expressed as an  $\mathbb{R}^n$ -valued function  $s \psi_s(s)$ , which we call the *heat-tension field*. According to this interpretation, it is natural to view the ‘geometric Littlewood-Paley projection’  $\psi_s$  as the main dynamic variable. More precisely, in order to establish an appropriate a priori estimate at the critical regularity for  $u$  (as in the Coulomb gauge case), we will analyze the dynamic equation obeyed by  $\psi_s$ . To make this scheme work, we need an analogue of the reconstruction formula, i.e., a means to reconstruct the harmonic map  $u$  and other variables of the derivative formulation such as  $\psi_\alpha$  and  $A_\alpha$  in terms of  $\psi_s$ .

This is exactly where Tao’s caloric gauge condition enters. As alluded to earlier, the caloric gauge condition corresponds to taking  $e(\infty) = e_\infty$  for a fixed constant frame  $e_\infty$  and parallel transporting this frame, i.e., imposing  $D_s e(s) = 0$  for  $0 < s < \infty$ . As a consequence, it follows that

$$\begin{aligned} \partial_s \psi_\alpha &= \mathbf{D}_\alpha \psi_s, & \partial_s A_\alpha &= \mathbf{R}(u(s))(\psi_s, \psi_\alpha) & \text{for } 0 < s < \infty \\ \psi_\alpha &\rightarrow 0, & A_\alpha &\rightarrow 0 & \text{as } s \rightarrow \infty \end{aligned} \quad (1.13)$$

Therefore,  $\psi_\alpha$  and  $A_\alpha$  can be recovered from  $\psi_s$  (under mild bootstrap assumptions) by simply integrating to  $s$  from  $\infty$ .

It is now clear how the caloric gauge resolves the difficulties faced by the Coulomb gauge. The main dynamic variable is now  $\psi_s$  instead of  $\psi_\alpha$ , which is evidently scalar. Indeed,  $\psi_s$  obeys a nonlinear wave equation<sup>3</sup> (3.96) with the scalar d’Alembertian as the principal part, for which global Strichartz estimates are known [2, 48]. From  $\psi_s$ , the other variables  $\psi_\alpha$  and  $A_\alpha$  are recovered by a simple integration from  $s = \infty$ . In effect, elliptic theory for the tensor  $A_\alpha$  in the Coulomb gauge is replaced by well-posedness theory for the harmonic map heat flow, which only relies on parabolic theory for the scalar heat equation.

*Further discussion on the caloric gauge.* So far we have emphasized the *scalar* nature of the main dynamic variable  $\psi_s$  as a key advantage of the caloric gauge over the Coulomb gauge, whose effectiveness becomes evident on a curved background such as ours.

Another important feature of the caloric gauge, applicable to any background, is that it is well-adapted to the large data setting. The construction of the caloric

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<sup>3</sup>We remark that the nonlinearity of this equation involves  $\psi_s$  at different  $s$ , resembling the equation satisfied by standard Littlewood-Paley projection of  $u$  in, say, the extrinsic formulation (1.5). The difference, of course, is that the nonlinearity for  $\psi_s$  enjoys more favorable structure.

gauge relies on the harmonic map heat flow, whose large data theory is well-understood in settings where one may hope to prove a large data result for the wave map. For further discussion, see Smith [61], where the caloric gauge was constructed for any time-dependent map on  $\mathbb{R} \times \mathbb{R}^2$  with sub-ground state energy. We also note the work [52, 53] of the second author on the extension of the caloric gauge to more general gauge theories by replacing the harmonic map heat flow by the Yang-Mills heat flow, where the key advantage over the Coulomb gauge seems to be applicability in the large data setting.

Finally, as already highlighted by Tao [70, 71], perhaps the deepest advantage of the caloric gauge is that it leads to a more favorable nonlinearity than the Coulomb gauge. More precisely, the worst interaction of  $\psi_\alpha$  in the equation for  $A$  in the Coulomb gauge, namely the high-high to low interaction, does not occur in the caloric gauge when suitably interpreted using the Littlewood-Paley analogy above<sup>4</sup>. This point was used crucially in the works [8, 19, 61] on Schrödinger maps, where such an extra cancellation appears to be essential. In our paper, this improved structure is reflected by the sufficiency of the usual Strichartz estimates to close the estimates; in particular, refined Strichartz estimates as in [51, 58] are not used.

*Remark 4.* Another idea that played a key role in the understanding the wave maps is that of null structure, introduced by Klainerman and Machedon [28] in this context, which refers to a special cancellation structure in the nonlinearity of the wave maps equation. We note that the null structure was also observed under symmetry in the work [13]. In this paper, the null structure of the wave maps does not play any role, since we work in high dimensions where the linear wave equation enjoys strong enough dispersion. Nevertheless, the null structure of the wave maps equation is expected to be essential in the case  $\Sigma = \mathbb{H}^2$ , which is the most interesting case.

**1.2. History and motivation.** The Cauchy problem for wave maps arises as a model problem in both general relativity, see e.g., [76], and in particle physics, see e.g., [21, 47], where it is called a nonlinear  $\sigma$ -model. From a purely mathematical point of view, the wave maps equation is perhaps the simplest and most natural geometric wave equation – the nonlinearity arises from the geometry of the target manifold – and is the hyperbolic analogue of the elliptic harmonic maps equation and the parabolic harmonic map heat flow equation.

*History.* At this point there is quite a lot known about the critical Cauchy problem for wave maps on  $\mathbb{R}_{t,x}^{1+d}$ , with a robust understanding in the very interesting energy critical dimension  $d = 2$ . To start, a detailed critical well-posedness theory was given in the classical works by Christodoulou, Tahvildar-Zadeh [12, 13] for spherically symmetric wave maps and by Shatah, Tahvildar-Zadeh [59, 60] for equivariant wave maps. Previously, Shatah [57] proved the existence of self-similar blow up solutions for energy supercritical wave maps into the 3-sphere.

Outside of spherical symmetry the sharp subcritical local well-posedness theory was developed by Klainerman, Machedon [28–30] and Klainerman, Selberg [33, 34] by making essential use of the *null form* structure in the nonlinearity, see Remark 4. The very difficult critical small data problem was partially settled by Tataru [72],

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<sup>4</sup>More precisely, (1.13) implies that  $A_\alpha(s)$  can be recovered just from  $s'\psi_s(s')$  at larger  $s'$ , which corresponds to lower frequency projections according to the Littlewood-Paley analogy discussed above.

who proved global well-posedness for data with small critical Besov norm  $\dot{B}_1^{\frac{d}{2},2} \times \dot{B}_1^{\frac{d}{2}-1,2}$ , and then completed by Tao [68, 69] for initial data that is small in  $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}$  for targets  $\mathcal{N} = \mathbb{S}^{N-1}$ . The seminal Tataru, Tao theory of small data wave maps was then extended to more general targets by Krieger [35, 36], Klainerman, Rodnianski [31], Tataru [74], Shatah, Struwe [58], and Nahmod, Stefanov, and Uhlenbeck [51]; see Section 1.1 for more on these works.

The bubbling analysis of Struwe [67] linked singularity formation for energy critical equivariant wave maps to the presence of nontrivial finite energy harmonic maps. This made apparent the decisive role that the geometry of the target plays in the energy critical problem. Later, explicit blow up constructions demonstrating the bubbling of a harmonic map were achieved in breakthrough works of Krieger, Schlag, Tataru [38], Raphaël, Rodnianski [54], and Rodnianski, Sterbenz [55] in the positively curved case  $\mathcal{N} = \mathbb{S}^2$ . Since there are no nontrivial finite energy harmonic maps from  $\mathbb{R}^2$  into negatively curved manifolds it was long conjectured that all smooth, decaying finite energy data lead to a global and scattering wave map evolution for maps  $\mathbb{R}_{t,x}^{1+2} \rightarrow \mathcal{N}$  in the case that all the sectional curvatures of  $\mathcal{N}$  are negative. This theorem was proved in 2009 in three remarkable and independent works by Krieger, Schlag [37], Sterbenz, Tataru [64, 65], and Tao [71]. Note that Krieger and Schlag [37] also developed a ‘twisted’ nonlinear profile decomposition in the spirit of the work of Bahouri and Gérard [3]. We also remark that in [64, 65] Sterbenz and Tataru proved the more general Threshold Theorem, which states that any smooth data with energy less than that of the minimal energy nontrivial harmonic map lead to a globally regular and scattering solution. For a further refinement of the threshold theorem, see [40], and for more detailed study of singularity formation for equivariant wave maps, see [9, 14–16, 27].

There are fewer works concerning wave maps on curved domains. Shatah, Tahvildar-Zadeh showed the existence and orbital stability of equivariant time-periodic wave maps,  $\mathbb{R} \times \mathbb{S}^2 \rightarrow \mathbb{S}^2$ . This was extended by the third author [56] to allow for maps  $\mathbb{R} \times \Sigma \rightarrow \mathbb{S}^2$  where  $\Sigma$  is diffeomorphic to  $\mathbb{S}^2$  and admits an  $SO(2)$  action. The first author established a critical small data global theory for wave maps on small asymptotically flat perturbations of  $\mathbb{R}^4$  [39] using the linear estimates of Metcalfe, Tataru [48]. Also, we note the recent work of D’Ancona, Zhang [18] who proved global existence for small equivariant wave maps on rotationally symmetric spacetimes for  $d \geq 3$ .

*Motivation.* There are several reasons to consider the wave maps equation on a curved spacetime  $\mathcal{M}$  as in (1.4). For one, note that such a wave map arises as a nonlinear model for Einstein’s equations. To be more specific, the Ernst potential of a Killing vector field  $X$  on a 1 + 3 dimensional Einstein-vacuum manifold  $\mathcal{M}$  can be viewed as a wave map  $\Phi : \widetilde{\mathcal{M}} \rightarrow \mathbb{H}^2$ , where  $\widetilde{\mathcal{M}} = \mathcal{M}/X$  is now a 1 + 2 dimensional spacetime; see for example [76] and the recent work of Ionescu and Klainerman [24], who addressed the stability of the Kerr Ernst potential. We also refer the reader to the recent work of Andersson, Gudapati, and Szeftel [1], who considered the wave map-Einstein system in an equivariant setting, and also to the work of Luk [46] who proved a global existence result for (3+1)-dimensional semilinear equations with null form structure on a Kerr spacetime.

Our main motivation for introducing curvature to the domain is that we view the treatment of wave maps on curved backgrounds as a natural extension of the



detailed theory on flat spacetime. A natural starting point is the following question: what effects does the geometry of the domain have on the global dynamics of the wave maps problem, especially for large data? In other words, are there new *nonlinear* phenomena that arise in the curved setting? Surely, one can affect the dynamics of the wave map equation by say, studying the problem on a compact domain or on manifolds  $\mathcal{M}$  that allow for the trapping of bicharacteristic rays. However, many of the most pressing issues that arise in such settings are already present at the level of the scalar linear wave equation on  $\mathcal{M}$ . There is a large body of literature devoted to the long time dispersive behavior of linear waves on curved spacetimes, and many interesting open questions remain; see for example [48] for one interesting direction.

To narrow the focus on the interplay between the curved geometry of the domain and the nonlinear structure of the wave maps equation, one can consider the constant sectional curvature model  $\Sigma = \mathbb{H}^d$  as a natural starting place. The global dispersive theory of linear waves  $\square_{\mathbb{H}^d} v = 0$  is now well understood, and in fact dispersion is stronger than for free waves on  $\mathbb{R}^{1+d}$  due to the exponential volume growth of balls in  $\mathbb{H}^d$ ; see [2, 4, 26, 49, 50] and [5–7, 25] for nonlinear applications.

To view the small data result in Theorem 1.1 (or more precisely, Theorem 4.1) and the techniques developed in this paper in this broader context, let us first describe our recent work on the energy critical wave maps equation on hyperbolic space, and the interesting nonlinear dynamics that arise in 2-dimensions. In [41, 42, 44] we considered *equivariant* wave maps from  $\mathbb{R} \times \mathbb{H}^2$  into two model targets, the  $2d$  hyperbolic space,  $\mathcal{N} = \mathbb{H}^2$  and the 2-sphere,  $\mathcal{N} = \mathbb{S}^2$ . In these settings the 1-equivariant wave map equation can be expressed entirely in terms of the polar coordinate  $\psi$  on the target, that is,  $u(t, r, \theta) = (\psi(t, r), \theta)$  and (1.3) reduces to

$$\psi_{tt} - \frac{1}{\sinh r} \partial_r (\sinh r \partial_r \psi) + \frac{g(\psi)g'(\psi)}{\sinh^2 r} = 0 \quad (1.14)$$

Here the target is endowed with the metric  $ds^2 = d\psi^2 + g^2(\psi)d\theta^2$  where  $g(\psi) = \sinh \psi$  in the case  $\mathcal{N} = \mathbb{H}^2$  and  $g(\psi) = \sin \psi$  if  $\mathcal{N} = \mathbb{S}^2$ .

By combining two well-known geometric facts – the conformal invariance of  $2d$  harmonic maps and their energy, and the conformal equivalence between  $\mathbb{H}^2$  and the unit disc  $\mathbb{D}^2$  – one finds explicit continuous 1-parameter families of finite energy harmonic maps for both targets  $\mathcal{N} = \mathbb{H}^2$  and  $\mathcal{N} = \mathbb{S}^2$ .

In the case of the target  $\mathcal{N} = \mathbb{H}^2$  the family of finite energy harmonic maps and their energies  $\mathcal{E}(P_\lambda, 0)$  are

$$P_\lambda(r) := 2\operatorname{arctanh}(\lambda \operatorname{arctanh}(r/2)), \quad \lambda \in [0, 1), \quad \mathcal{E}(P_\lambda, 0) = 4\pi \frac{\lambda^2}{1 - \lambda^2}. \quad (1.15)$$

The very existence of these maps is in stark contrast to the corresponding Euclidean problem; there are no finite energy harmonic maps  $\mathbb{R}^2 \rightarrow \mathbb{H}^2$  due to the well-known Eells-Sampson theory [20]. In [41] we proved that every  $P_\lambda$  is asymptotically stable under small finite energy equivariant perturbations. Moreover, there can be no finite time blow up for wave maps  $\mathbb{R} \times \mathbb{H}^2 \rightarrow \mathbb{H}^2$  since otherwise, by the same Struwe bubbling analysis as above, there would have to exist a nontrivial finite energy Euclidean harmonic map into  $\mathbb{H}^2$ , yielding a contradiction. Next, note that the harmonic map  $P_\lambda(r)$  satisfies

$$P_\lambda(r) \rightarrow 2\operatorname{arctanh}(\lambda) \quad \text{as } r \rightarrow \infty.$$

In fact, the space of equivariant finite energy data naturally splits into disjoint classes  $\mathcal{E}_\lambda$  for  $\lambda \in [0, 1]$  where each  $(\psi_0, \psi_1) \in \mathcal{E}_\lambda$  is characterized by  $\psi_0(r) \rightarrow 2\operatorname{arctanh}(\lambda)$  as  $r \rightarrow \infty$ . Moreover, the classes  $\mathcal{E}_\lambda$  are preserved by the smooth wave map flow and  $(P_\lambda, 0)$  uniquely minimizes the energy in  $\mathcal{E}_\lambda$ ; see [41, 42]. One is led to the following conjecture regarding the asymptotic behavior of solutions in the equivariant setting.

**Conjecture 1** (Soliton resolution for equivariant wave maps  $\mathbb{R} \times \mathbb{H}^2 \rightarrow \mathbb{H}^2$ ). *Consider the Cauchy problem (1.14) with finite energy initial data  $(\psi_0, \psi_1)$ . Let*

$$\lambda := \tanh(\psi_0(\infty)/2) \in [0, 1].$$

*Then (1.14) is globally well-posed, and the solution scatters to  $P_\lambda$  as  $t \rightarrow \pm\infty$ .*

In [42] the conjecture was verified for all data with endpoints  $\lambda \leq \Lambda_0$  where  $\Lambda_0 \geq 0.57$ , leaving open the range of endpoints  $\lambda \in [\Lambda_0, 1]$ .

More ambitiously, one can formulate the soliton resolution conjecture in the nonequivariant setting. In this case, using conformal equivalence of  $\mathbb{H}^2$  and  $\mathbb{D}^2$ , the space of finite energy maps splits into disjoint classes  $\mathcal{E}_{\partial u}$  parametrized by boundary data  $\partial u$  at spatial infinity that are consistent with finite energy. These classes are preserved by the wave map flow, and the minimizer  $\mathcal{P}_{\partial u}$  of the Dirichlet energy in each class is a harmonic map. The soliton resolution conjecture may then be phrased in terms  $\mathcal{P}_{\partial u}$  as in Conjecture 1.

An interesting milestone in this direction would be the global asymptotic stability of  $\mathcal{P}_\lambda(r, \theta) := (P_\lambda(r), \theta)$  under arbitrarily large non-equivariant perturbations, where  $\mathcal{P}_\lambda$  has the advantage of being explicit.

**Conjecture 2.** *Let  $(u_0, u_1)$  be smooth finite energy initial data for (1.4) for  $\mathcal{M} = \mathbb{R} \times \mathbb{H}^2$  and  $\mathcal{N} = \mathbb{H}^2$ . Let  $\lambda \in [0, 1]$  be fixed and assume that outside some compact set  $K \Subset \mathbb{H}^2$  we have*

$$u_0(x) = \mathcal{P}_\lambda(x) \quad \text{for } x \in \mathbb{H}^2 - K, \quad (1.16)$$

*Then the unique wave map evolution  $(u, \partial_t u)(t)$  associated to the data  $(u_0, u_1)$  is globally regular and scatters to  $(\mathcal{P}_\lambda, 0)$  as  $t \rightarrow \pm\infty$ .*

Aside from establishing the small data scattering theory in high dimensions, this paper can be viewed as a first step towards the non-equivariant theory for  $\Sigma = \mathbb{H}^2$ , and in particular Conjecture 2.

*Remark 5.* There are even more challenges and unexpected phenomena in the other case considered in [41], namely when  $\mathcal{N} = \mathbb{S}^2$ . The equivariant harmonic maps from  $\mathbb{H}^2 \rightarrow \mathbb{S}^2$  and their energies  $\mathcal{E}(Q_\lambda, 0)$  are given by

$$Q_\lambda(r) := 2 \arctan(\lambda \operatorname{arctanh}(r/2)), \quad \lambda \in [0, \infty), \quad \mathcal{E}(Q_\lambda, 0) = 4\pi\lambda^2(1 + \lambda^2)^{-1}. \quad (1.17)$$

The stability of  $Q_\lambda$  is investigated in [41]. We showed that there exists  $\lambda_0 > 1$  so that each  $Q_\lambda$  for  $\lambda < \lambda_0$  is asymptotically stable under small finite energy equivariant perturbations. However there is more interesting behavior for  $\lambda \gg 1$ . In particular, for  $\lambda$  large enough the Schrödinger operator  $\mathcal{L}_\lambda$  obtained by linearizing about  $Q_\lambda$  has an eigenvalue in its spectral gap  $(0, 1/4)$ . This gap eigenvalue leads to a solution to the linearized equation that does not decay as  $t \rightarrow \pm\infty$ . On the other hand, one can show that any small finite energy perturbation of  $Q_\lambda$  must lead to a globally regular nonlinear evolution. This is a consequence of the fact

that any finite time blow up in this energy critical, equivariant setting must occur by energy concentration at  $r = 0$  and thus the global geometry of the domain is irrelevant. In fact blow up can be characterized by the bubbling of a *Euclidean* harmonic map exactly as in [67], and this requires more energy than a sufficiently small perturbation of  $Q_\lambda$ ; see [42]. These observations suggest that in fact all of the  $Q_\lambda$  are asymptotically stable in the energy space, with the stability of the  $Q_\lambda$  for large  $\lambda$  occurring via a completely nonlinear mechanism, such as the *radiative damping* described in the work of Soffer, Weinstein [62].

## 2. GEOMETRIC AND ANALYTIC PRELIMINARIES

**2.1. Notation.** We will try to maintain the following conventions for indices: We reserve greek lowercase indices  $\alpha, \beta, \gamma, \dots \in \{0, 1, 2, 3, 4\}$  for coordinates on  $\mathcal{M} = \mathbb{R} \times \mathbb{H}^4$ . We use Latin indices  $a, b, c, \dots \in \{1, 2, 3, 4\}$  when restricting to coordinates on  $\mathbb{H}^4$ . These indices are raised and lowered using the appropriate domain metrics introduced below. Sometimes we will use the bold letters  $\mathbf{a}, \mathbf{b}, \dots$  for coordinates  $(s, t, x^1, \dots, x^4)$  on  $\mathbb{R}^+ \times \mathcal{M}$ . Next, we will use lowercase Latin indices beginning at  $i$ , namely  $i, j, k, \ell, m$  for components in the tangent space of  $\mathcal{N}$ ; hence they run from  $1, \dots, n$  where  $n$  is the dimension of  $\mathcal{N}$ . The capital letters  $I, J, K, L$  will be used for indices in  $\mathbb{R}^N$ , which arises as the ambient space to which  $\mathcal{N}$  is embedded. Finally, throughout the paper we adopt the standard convention of silently summing over repeated upper and lower indices.

**2.2. Geometry of the domain.** Let  $\mathbb{R}^{d+1}$  denote the  $(d+1)$ -dimensional Minkowski space with rectilinear coordinates  $\{y^1, \dots, y^d, y^0\}$  and metric  $\mathbf{m}$  given in these coordinates by  $\mathbf{m} = \text{diag}(1, \dots, 1, -1)$ . One can then define the  $d$ -dimensional hyperbolic space  $\mathbb{H}^d$  as

$$\mathbb{H}^d := \{y \in \mathbb{R}^{d+1} \mid y^0 = 1, (y^0)^2 - \sum_{a=1}^d (y^a)^2 = 1, y^0 > 0\}$$

The Riemannian structure on  $\mathbb{H}^d$  is obtained by pulling back the metric  $\mathbf{m}$  on  $\mathbb{R}^{d+1}$  via the inclusion map  $\iota : \mathbb{H}^d \hookrightarrow \mathbb{R}^{d+1}$ . In particular the Riemannian metric  $\mathbf{h}$  on  $\mathbb{H}^d$  is given by  $\mathbf{h} = \iota^* \mathbf{m}$ .

We note that  $\mathbb{H}^d$  admits global coordinate systems. Given a global coordinate system  $\{x^a\}$  on  $\mathbb{H}^d$  denote by  $\mathbf{h}_{ab}$  the components of  $\mathbf{h}$  and  $\mathbf{h}^{ab}$  the components of the inverse matrix  $\mathbf{h}^{-1}$ , i.e.,

$$\mathbf{h}_{ab} := \mathbf{h} \left( \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right), \quad \mathbf{h}^{ab} := \mathbf{h}^{-1}(dx^a, dx^b)$$

The metric  $\boldsymbol{\eta} = (\eta_{\alpha\beta})$  on the Lorentzian manifold  $\mathcal{M} = \mathbb{R} \times \mathbb{H}^d$  is then given in coordinates by  $\eta_{00} = -1$ ,  $\eta_{0a} = 0$  and  $\eta_{ab} = \mathbf{h}_{ab}$ . We denote the Christoffel symbols on  $\mathbb{R} \times \mathbb{H}^d$  by

$$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2} \boldsymbol{\eta}^{\gamma\mu} (\partial_\alpha \eta_{\mu\beta} + \partial_\beta \eta_{\alpha\mu} - \partial_\mu \eta_{\alpha\beta})$$

Given a vector field  $X = X^\alpha \frac{\partial}{\partial x^\alpha}$  in  $\Gamma(T\mathcal{M})$  or a 1-form  $\omega = \omega_\alpha dx^\alpha \in \Gamma(T^*\mathcal{M})$  the metric covariant derivative  $\nabla$  is defined in coordinates by

$$\nabla_\alpha X = (\partial_\alpha X^\beta + \Gamma_{\alpha\gamma}^\beta X^\gamma) \partial_\beta, \quad \nabla_\alpha \omega = (\partial_\alpha \omega_\beta - \Gamma_{\alpha\gamma}^\gamma \omega_\gamma) dx^\beta$$

From these definitions one can generalize the definition of  $\nabla$  to  $(p, q)$ -tensor fields of arbitrary rank by requiring that:  $\nabla_\alpha f = \partial_\alpha f$  for  $(0, 0)$ -tensors (i.e., functions),  $\nabla_\alpha$  agrees with the preceding definition for  $(1, 0)$ - or  $(0, 1)$ -tensors (i.e., vector fields or 1-forms) and  $\nabla$  obeys the Leibniz rule with respect to tensor products. We note that  $\nabla_0 = \partial_t$  (i.e.,  $\Gamma_{0\beta}^\alpha = 0$ ) and that  $\nabla_a$  for a Latin index  $a$  will denote a covariant derivative over  $\mathbb{H}^d$ . Next, we will denote by  $\mathcal{R}$  the Riemann curvature tensor on  $\mathbb{H}^d$ . Recall that for a vector field  $\xi^a \frac{\partial}{\partial x^a}$  in  $\Gamma(T\mathbb{H}^d)$  we have the formula

$$[\nabla_a, \nabla_b]\xi^c = \mathcal{R}_{ab}^c \xi^e \quad (2.1)$$

as well as

$$[\nabla_a, \nabla_b]\xi^a = \mathcal{R}_{cab}^a \xi^c = \mathcal{R}_{cb} \xi^c \quad (2.2)$$

where  $\mathcal{R}_{cb}$  denote the components of the Ricci curvature tensor. Since  $\mathbb{H}^d$  has constant sectional curvature  $\mathcal{K} = -1$  we note that

$$\mathcal{R}_{ab} = -(d-1)\mathbf{h}_{ab} \quad (2.3)$$

Finally, we observe that the Riemann curvature tensor on  $\mathcal{M} = \mathbb{R} \times \mathbb{H}^d$  vanishes when contracted with  $\partial_t$  in any slot, and it agrees with  $\mathcal{R}_{cab}^a$  when contracted with vectors tangent to  $\mathbb{H}^d$ . For this reason, we will slightly abuse notation by referring to the Riemann curvature tensor on  $\mathcal{M}$  also as  $\mathcal{R}$ .

In Section 3 and beyond, we will work exclusively in the case  $d = 4$ .

**2.3. Geometry of maps  $u : \mathcal{M} \rightarrow \mathcal{N}$ .** As we saw in the introduction the wave maps equation (1.3) is defined using the geometric structure of the pullback vector bundle  $u^*T\mathcal{N}$  for maps  $u : \mathcal{M} \rightarrow \mathcal{N}$ . In this subsection, we develop the differential-geometric formalism for analysis on the bundle  $u^*T\mathcal{N}$ .

Let  $\mathcal{M}$  be a smooth manifold equipped with a torsion-free connection  $\nabla$  on the tensor bundle, and assume furthermore that  $\mathcal{M}$  is contractible. A specific example to have in mind is of course  $\mathcal{M} = \mathbb{R} \times \mathbb{H}^4$  with  $\nabla$  the Levi-Civita connection. Such a general formulation will be useful later in Section 3, where we consider the harmonic map heat flow and construct the caloric gauge.

Let  $u$  be a smooth map from  $\mathcal{M}$  to  $\mathcal{N}$ , where  $\mathcal{N}$  obeys the hypothesis of Theorem 1.1. The pullback bundle  $u^*T\mathcal{N}$  is the vector bundle over  $\mathcal{M}$  whose fiber at  $p \in \mathcal{M}$  is the tangent space  $T_{u(p)}\mathcal{N}$ . Smooth sections of  $u^*T\mathcal{N}$  are maps  $\phi : \mathcal{M} \rightarrow T\mathcal{N}$  so that  $\phi(p) \in T_{u(p)}\mathcal{N}$  for each  $p \in \mathcal{M}$ . The Riemannian connection on  $T\mathcal{N}$  induces the pullback covariant derivative  $D$  on  $u^*T\mathcal{N}$ . In local coordinates  $\{x^\alpha\}_{\alpha=0}^d$  on  $\mathcal{M}$  and  $\{u^j\}_{j=1}^n$  on  $\mathcal{N}$  we have

$$D_\alpha \phi = (\partial_\alpha \phi^j + {}^{(\mathcal{N})}\Gamma_{ik}^j(u) \phi^i \partial_\alpha u^k) \frac{\partial}{\partial u^j} \quad (2.4)$$

where  ${}^{(\mathcal{N})}\Gamma_{ik}^j$  are the Christoffel symbols on  $T\mathcal{N}$ . The pullback covariant derivative  $D_\alpha$  is naturally extended to  $u^*T\mathcal{N}$ -valued tensors (i.e., sections of  $T^{\otimes p}\mathcal{M} \otimes (T^*)^{\otimes q}\mathcal{M} \otimes u^*T\mathcal{N}$ ) using the connection  $\nabla_\alpha$  on the tensor bundle of  $\mathcal{M}$ .

We will study the wave maps system by expressing the equation satisfied by the differential  $du$  in terms of a global moving frame (or in short, simply a *frame*) on  $u^*T\mathcal{N}$ . This refers to a collection  $e = (e_1, \dots, e_n)$  of global (i.e., defined on all of  $\mathcal{M}$ ) sections of  $u^*T\mathcal{N}$  that form an orthonormal basis of  $T_{u(p)}\mathcal{M}$  on each  $p \in \mathcal{M}$ . By the assumption that the base manifold  $\mathcal{M}$  is contractible, at least one such

a frame exists. Expressing each fiber  $T_{u(p)}\mathcal{N}$  in terms of the orthonormal basis  $(e_1, \dots, e_n)(p)$ , we see that  $e$  defines a global trivialization of  $u^*T\mathcal{N}$ , i.e.,

$$e : E_n \rightarrow T\mathcal{N}, \quad v = (p; v^1, \dots, v^n) \mapsto ev = e_k(p)v^k$$

where  $E_n$  refers to the trivial bundle  $E_n = \mathcal{M} \times \mathbb{R}^n$ . Moreover,  $e$  induces an isomorphism between the spaces of sections

$$e : \Gamma(E_n) \rightarrow \Gamma(u^*T\mathcal{N}), \quad \varphi = (\varphi^1, \dots, \varphi^n) \mapsto e\varphi = e_k\varphi^k$$

Note that  $\Gamma(E_n)$  is simply the space of  $\mathbb{R}^n$ -valued functions on  $\mathcal{M}$ . This isomorphism extends naturally to  $u^*T\mathcal{N}$ -valued tensors; we make a slight abuse of notation and refer to all these isomorphisms by  $e$ .

Given a map  $B : \mathcal{M} \rightarrow SO(n)$  and a frame  $e$ , we may obtain a new frame  $e'$  by setting

$$e \mapsto eB =: e', \quad \text{or } e'_j := e_i B_j^i \quad (2.5)$$

Conversely, observe that any two frames  $e, e'$  with the same orientation must be related to each other by  $e' = eB$  for some  $B : \mathcal{M} \rightarrow SO(n)$ . In other words, the wave maps system in the global moving frame formulation has an  $SO(n)$  gauge structure.

Let  $e$  be a frame. Associated to the map  $u$  we have the differential  $du$ , which is a  $u^*T\mathcal{N}$ -valued 1-form. We denote by  $\Psi(\partial_\alpha) = \psi_\alpha$  the description of  $du$  with respect to  $e$ ,

$$du = e\Psi, \quad \partial_\alpha u = e\psi_\alpha \quad (2.6)$$

The pullback covariant derivative  $D_\alpha$  admits a matrix description in the global trivialization  $E_n$ . We first introduce a connection  $\mathbf{D}_\alpha$  on  $E_n$  induced by  $D_\alpha$ , characterized by the identity

$$D_\alpha e\varphi = e\mathbf{D}_\alpha\varphi \quad (2.7)$$

Using the connection  $\nabla$  on  $E_n$  (defined component-wisely) as a reference, we may express  $\mathbf{D}$  in the form

$$\mathbf{D} = \nabla + A \quad (2.8)$$

where  $A$  is an  $\mathfrak{so}(n)$ -valued (i.e., anti-symmetric traceless matrix) 1-form on  $\mathcal{M}$ . The latter fact is evident from the following alternative characterization of  $A$ :

$$A = e^{-1}De, \quad \text{or } A_{j,\alpha}^i = \langle D_\alpha e_j, e_i \rangle$$

We call  $A$  the *connection 1-form* associated to  $\mathbf{D}$ . Going back to  $du = e\Psi$ , by the torsion-free property inherited from both the domain and target connections, note that

$$\mathbf{D}_\alpha\psi_\beta = \mathbf{D}_\beta\psi_\alpha \quad (2.9)$$

If  $u$  is a wave map and  $e\psi_\alpha$  is the representation of  $\partial_\alpha u$  in the frame  $e$ , the wave maps equation can be rewritten succinctly as

$$\eta^{\alpha\beta}(\mathbf{D}_\alpha\psi)_\beta^j = 0 \quad (2.10)$$

or in coordinate free notation

$$\text{tr}_\eta \mathbf{D}\Psi = 0 \quad (2.11)$$

Next, we introduce the *curvature 2-form* associated to  $\mathbf{D}$ , which is defined to be the commutator

$$F_{\alpha\beta}\varphi = (\mathbf{D}_\alpha\mathbf{D}_\beta - \mathbf{D}_\beta\mathbf{D}_\alpha)\varphi \quad (2.12)$$

Note that  $F$  is an  $\mathfrak{so}(n)$ -valued 2-form on  $\mathcal{M}$ . Using (2.8),  $F$  can be expressed in terms of  $A$  as

$$F = dA + A \wedge A, \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] \quad (2.13)$$

The curvature 2-form  $F$  is related through  $u$  to the Riemann curvature  $R$  of the target manifold  $\mathcal{N}$  as follows:

$$eF_{\alpha\beta}\varphi = R(u)(\partial_\alpha u, \partial_\beta u)e\varphi = R(u)(e\psi_\alpha, e\psi_\beta)e\varphi \quad (2.14)$$

In the sequel we will have to perform changes of frame as in (2.5) and we record here the effect of a change of basis on the relevant objects,  $F, A, \psi$ . For  $B : \mathcal{M} \rightarrow SO(n)$  we have the *gauge symmetry* relative to the map  $u : \mathcal{M} \rightarrow \mathcal{N}$ :

$$\begin{aligned} u &\mapsto u; & e &\mapsto eB; & \psi_\alpha &\mapsto B^{-1}\psi_\alpha; \\ F_{\alpha\beta} &\mapsto B^{-1}F_{\alpha\beta}B; & A_\alpha &\mapsto B^{-1}\partial_\alpha B + B^{-1}A_\alpha B \end{aligned} \quad (2.15)$$

Finally, we make a simple remark on the computation of the commutator  $[\mathbf{D}_\alpha, \mathbf{D}_\beta]$ . For a section  $\varphi \in \Gamma(E_n)$ , this commutator is precisely the matrix-valued 2-form  $F_{\alpha\beta}$ . For a  $\mathbb{R}^n$ -valued tensor  $T = T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}$ , the domain curvature tensor  $\mathcal{R}$  associated to  $\nabla$  arises, i.e.,

$$\begin{aligned} [\mathbf{D}_\alpha, \mathbf{D}_\beta]T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} &= F_{\alpha\beta}T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} \\ &+ \mathcal{R}^{\mu_1}_{\gamma\alpha\beta}T^{\gamma \dots \mu_p}_{\nu_1 \dots \nu_q} + \dots + \mathcal{R}_{\nu_q}^{\gamma}_{\alpha\beta}T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \gamma} \end{aligned} \quad (2.16)$$

**2.3.1. Extrinsic representation.** Using the isometric embedding  $\mathcal{N} \hookrightarrow \mathbb{R}^N$ , the map  $u$  admits a representation as an  $\mathbb{R}^N$ -valued map  $u = (u^K)_{K=1, \dots, N}$ , which will be referred to as the *extrinsic representation*. This point of view is convenient for analysis when the delicate gauge structure described in (2.15) need not be exploited. See, for instance, our analysis of the harmonic map heat flow in Section 3.

In this viewpoint, any section  $\phi$  of  $u^*T\mathcal{N}$  admits a representation as an  $\mathbb{R}^N$ -valued function on  $\mathcal{M}$ . Since  $\mathcal{N} \hookrightarrow \mathbb{R}^N$  is isometric, we have

$$\left( \sum_{K=1}^N |\phi^K|^2 \right)^{1/2} = |\phi|_{u^*T\mathcal{N}} = |\phi|_{\mathbb{R}^N} \quad (2.17)$$

where  $e\varphi = \phi$ . We will abuse the notation a bit and write  $|\phi|_{\mathbb{R}^N}$  as a shorthand for the left-hand side. In particular, the differential  $du$  is represented by an  $\mathbb{R}^N$ -valued 1-form  $du = (du^K)_{K=1, \dots, N}$  on  $\mathcal{M}$  and we have

$$|du(X)|_{\mathbb{R}^N} = |du(X)|_{u^*T\mathcal{N}} = |\Psi(X)|_{\mathbb{R}^n} \quad (2.18)$$

for any vector  $X$  tangent to  $\mathcal{M}$ . The pullback covariant derivative  $D$  on  $u^*T\mathcal{N}$  has the following natural interpretation: Given a section  $\phi$  of  $u^*T\mathcal{N}$ , its covariant derivatives  $D_\alpha \phi$  over a point  $x \in \mathcal{M}$  are the standard coordinate derivatives  $\partial_\alpha \phi$  of  $\phi$  projected orthogonally onto  $T_{u(x)}\mathcal{N}$ . This leads to the formula

$$D_\alpha \phi^K = \partial_\alpha \phi^K + S_{IJ}^K(u) \phi^I \partial_\alpha u^J \quad (2.19)$$

where  $S$  is the second fundamental form.

**2.4. Differentiation of the curvature 2-form.** Let  $u : \mathcal{M} \rightarrow \mathcal{N}$  be a map and  $e$  be a given frame. Let  $F$  be the curvature 2-form of  $\mathbf{D}$  defined above in (2.12). We introduce the covariant derivative of  $F$  by the formula

$$\mathbf{D}_\mu F_{\alpha\beta} := \nabla_\mu F_{\alpha\beta} + [A_\mu, F_{\alpha\beta}], \quad (2.20)$$

where  $\nabla_\mu$  is to be understood in the component-wise sense. This definition is natural<sup>5</sup> in view of the gauge symmetry of  $F$  in (2.15). Observe also that  $\mathbf{D}F$  is characterized by the Leibniz rule

$$\mathbf{D}(F\varphi) = (\mathbf{D}F)\varphi + F\mathbf{D}\varphi, \quad (2.21)$$

for any  $\mathbb{R}^n$ -valued function  $\varphi$ .

Our goal now is to give formulae for differentiating  $F$ . Recall that  $F$  is related to the Riemann curvature  $R$  on  $\mathcal{N}$  by the formula (2.14). To have a description of  $F$  entirely in terms of objects defined relative to the frame  $e$ , we introduce the pullback curvature  $\mathbf{R}$  by

$$e\mathbf{R}(\psi^{(1)}, \psi^{(2)})\varphi = R(u)(e\psi^{(1)}, e\psi^{(2)})e\varphi$$

for  $\mathbb{R}^n$ -valued functions  $\psi^{(j)}$  ( $j = 1, 2$ ) and  $\varphi$ . Observe that  $\mathbf{R}(\psi^{(1)}, \psi^{(2)})$  takes values in  $\mathfrak{so}(n)$ . The identity (2.14) may be rewritten as

$$F_{\alpha\beta} = \mathbf{R}(\psi_\alpha, \psi_\beta). \quad (2.22)$$

To facilitate the computation of derivatives of  $F$ , we introduce the pullback differentiated curvature (for  $m = 1, 2, \dots$ )

$$\mathbf{R}^{(m)}(\psi^{(1)}, \dots, \psi^{(m)}; \varphi^{(1)}, \varphi^{(2)}) = (\nabla^{(m)} R)(u)(e\psi^{(1)}, \dots, e\psi^{(m)}; e\varphi^{(1)}, e\varphi^{(2)})e,$$

under the convention  $\nabla^{(m)} R(X^{(1)}, \dots, X^{(m)}; Y, Z) = \nabla_{X^{(1)} \dots X^{(m)}}^{(m)} R(Y, Z)$ . In the case  $m = 0$ , we take  $\mathbf{R}^{(0)} = \mathbf{R}$ . From the definition, we may immediately derive the differentiation formula

$$(\mathbf{D}_\mu \mathbf{R}^{(m)})(\psi^{(1)}, \dots, \psi^{(m)}; \varphi^{(1)}, \varphi^{(2)}) = \mathbf{R}^{(m+1)}(\psi_\mu, \psi^{(1)}, \dots, \psi^{(m)}; \varphi^{(1)}, \varphi^{(2)}) \quad (2.23)$$

Combined with the Leibniz rule, it is now straightforward to compute covariant derivatives of  $F$  of any order. For instance, in order to compute  $\mathbf{D}F$ , we first apply the Leibniz rule to obtain

$$\mathbf{D}_\mu F_{\alpha\beta} = (\mathbf{D}_\mu \mathbf{R}^{(0)})(\psi_\alpha, \psi_\beta) + \mathbf{R}^{(0)}(\mathbf{D}_\mu \psi_\alpha, \psi_\beta) + \mathbf{R}^{(0)}(\psi_\alpha, \mathbf{D}_\mu \psi_\beta)$$

Then by (2.23), we arrive at the formula

$$\mathbf{D}_\mu F_{\alpha\beta} = \mathbf{R}^{(1)}(\psi_\mu; \psi_\alpha, \psi_\beta) + \mathbf{R}^{(0)}(\mathbf{D}_\mu \psi_\alpha, \psi_\beta) + \mathbf{R}^{(0)}(\psi_\alpha, \mathbf{D}_\mu \psi_\beta) \quad (2.24)$$

Finally, by the bounded geometry assumptions, observe that the following point-wise bound holds for each  $\mathbf{R}^{(m)}$ :

$$\left| \mathbf{R}^{(m)}(\psi^{(1)}, \dots, \psi^{(m)}; \varphi^{(1)}, \varphi^{(2)}) \right| \lesssim \left| \psi^{(1)} \right| \dots \left| \psi^{(m)} \right| \left| \varphi^{(1)} \right| \left| \varphi^{(2)} \right| \quad (2.25)$$

where we use the Hilbert-Schmidt norm for matrices on the left-hand side.

---

<sup>5</sup>Geometrically,  $F$  is an  $\text{ad } P$ -valued 2-form, where  $\text{ad } P$  is the vector bundle with fiber  $\mathfrak{so}(n)$ , structure group  $SO(n)$  acting by the adjoint action  $(g, a) \mapsto gag^{-1}$ , and the same structure coefficients as  $u^*T\mathcal{N}$ . Then  $\mathbf{D}$  defined as above is simply the induced covariant derivative on  $\text{ad } P$ .

**2.5. Function Spaces on  $\mathbb{H}^d$ .** With the Riemannian structure we can define the relevant function spaces on  $\mathbb{H}^d$ . Let  $f : \mathbb{H}^d \rightarrow \mathbb{R}$  be a smooth function. The  $L^p(\mathbb{H}^d)$  spaces are defined for  $1 \leq p < \infty$  by

$$\|f\|_{L^p(\mathbb{H}^d)} := \left( \int_{\mathbb{H}^d} |f(x)|^p \, d\text{vol}_{\mathbf{h}} \right)^{\frac{1}{p}}, \quad \|f\|_{L^\infty(\mathbb{H}^d)} := \sup_{x \in \mathbb{H}^d} |f(x)|$$

The Sobolev norm  $W^{k,p}(\mathbb{H}^d; \mathbb{R})$  are defined as follows, see for example [22]:

$$\|f\|_{W^{k,p}(\mathbb{H}^d; \mathbb{R})} := \sum_{\ell=0}^k \left( \int_{\mathbb{H}^d} |\nabla^{(\ell)} f|^p \, d\text{vol}_{\mathbf{h}} \right)^{\frac{1}{p}} \quad (2.26)$$

where here  $\nabla^{(\ell)} f$  is the  $\ell$ th covariant derivative of  $f$  with the convention that  $\nabla^{(0)} f = f$ . We note that in coordinates the components of  $\nabla f$  are given by  $(\nabla f)_a = \partial_a f$ , and those of  $\nabla^{(2)} f$  are then

$$(\nabla^2 f)_{ab} = \partial_a \partial_b f - \Gamma_{ab}^c \partial_c f \quad (2.27)$$

and so on. We then have

$$|\nabla^{(\ell)} f|^p = \left( \mathbf{h}^{a_1 b_1} \dots \mathbf{h}^{a_\ell b_\ell} (\nabla^{(\ell)} f)_{a_1 \dots a_\ell} (\nabla^{(\ell)} f)_{b_1 \dots b_\ell} \right)^{\frac{p}{2}} \quad (2.28)$$

One then defines the Sobolev space  $W^{k,p}(\mathbb{H}^d; \mathbb{R})$  to be the completion of all smooth compactly supported functions  $f \in C_0^\infty(\mathbb{H}^d; \mathbb{R})$  under the norm (2.26); see [22, Theorem 2.8]. We note that for  $p = 2$  we write  $W^{k,2}(\mathbb{H}^d; \mathbb{R}) =: H^k(\mathbb{H}^d; \mathbb{R})$ . The Sobolev spaces  $W^{s,p}$  for  $s \geq 0$  are then defined by interpolation.

Alternatively one can define Sobolev spaces on  $\mathbb{H}^d$  using the spectral theory of the Laplacian  $-\Delta_{\mathbb{H}^d}$ , which we will now denote simply by  $-\Delta$ . Given  $s \in \mathbb{R}$ , the fractional Laplacian  $(-\Delta)^{\frac{s}{2}}$  is a well-defined operator on, say  $C_0^\infty(\mathbb{H}^d)$  via the spectral theory of  $-\Delta$ . Given a function  $f \in C_0^\infty(\mathbb{H}^d)$ , we set

$$\|f\|_{\widetilde{W}^{s,p}(\mathbb{H}^d)} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{H}^d)}$$

and define  $\widetilde{W}^{s,p}(\mathbb{H}^d)$  to be the completion of  $C_0^\infty(\mathbb{H}^d)$  under the norm above.

The fractional Laplacian  $(-\Delta)^{\frac{s}{2}}$  is bounded on  $L^p(\mathbb{H}^d)$  for all  $s \leq 0$  and all  $p \in (1, \infty)$ . Using this fact one can show that  $\widetilde{W}^{s,p} = W^{s,p}$  with equivalent norms, where the latter space is defined using the Riemannian structure as above; see for example [73] for a proof, and also Lemma 2.5 below. In the sequel we will often use these two definitions interchangeably, with expressions like

$$\|\nabla(-\Delta)^\ell f\|_{L^2} \simeq \|\nabla^{(2\ell+1)} f\|_{L^2} \simeq \|f\|_{H^{2\ell+1}} \quad (2.29)$$

where the last  $\simeq$  is due to the Poincaré inequality; see Lemma 2.1. We will often use the notation

$$\langle f | g \rangle_{L^2} := \int_{\mathbb{H}^d} f(x) g(x) \, d\text{vol}_{\mathbf{h}} \quad (2.30)$$

to denote the  $L^2(\mathbb{H}^d)$  inner product of  $f, g$ .

We state some basic functional inequalities that hold on  $\mathbb{H}^d$ .



**Lemma 2.1** (Poincaré inequality). *Let  $f \in C_0^\infty(\mathbb{H}^d)$ . Then the following inequalities hold.*

$$\|f\|_{L^2} \leq \left(\frac{d-1}{2}\right)^{-1} \|\nabla f\|_{L^2} \quad (2.31)$$

$$\|f\|_{L^2} \leq \left(\frac{d-1}{2}\right)^{-2} \|\Delta f\|_{L^2} \quad (2.32)$$

Both inequalities are equivalent to the well-known fact that  $-\Delta$  on  $\mathbb{H}^d$  has a spectral gap of  $(\frac{d-1}{2})^2$  (see [10]).

**Lemma 2.2** (Gagliardo-Nirenberg inequality). *Let  $f \in C_0^\infty(\mathbb{H}^d)$ . Then for any  $1 < p < \infty$ ,  $p \leq q \leq \infty$  and  $0 < \theta < 1$  such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{\theta}{d}$$

*we have*

$$\|f\|_{L^q(\mathbb{H}^d)} \lesssim \|f\|_{L^p(\mathbb{H}^d)}^{1-\theta} \|\nabla f\|_{L^p(\mathbb{H}^d)}^\theta \quad (2.33)$$

This lemma can be proved using Littlewood-Paley projections defined (say) by the linear heat equation (see [25, 44]). In Lemma 2.2, we note that  $\|\nabla f\|_{L^p} \simeq \|(-\Delta)^{\frac{1}{2}} f\|_{L^p}$  (see [73]). We omit the routine details.

**Lemma 2.3** (Sobolev embedding). *Let  $f \in C_0^\infty(\mathbb{H}^d)$ . Then for any  $1 < p < d$  and  $p \leq q < \infty$  with*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$$

*we have*

$$\|f\|_{L^q(\mathbb{H}^d)} \lesssim \|\nabla f\|_{L^p(\mathbb{H}^d)} \quad (2.34)$$

For a proof, we refer to [22, Theorem 3.2].

We conclude with some statements related to the fractional Laplacian  $(-\Delta)^\alpha$ .

**Lemma 2.4** ( $L^2$  interpolation inequalities). *Let  $f \in C_0^\infty(\mathbb{H}^d)$ . Then for  $0 \leq \beta \leq \alpha$  we have*

$$\|(-\Delta)^\beta f\|_{L^2(\mathbb{H}^d)} \lesssim \|f\|_{L^2(\mathbb{H}^d)}^{1-\frac{\beta}{\alpha}} \|(-\Delta)^\alpha f\|_{L^2(\mathbb{H}^d)}^{\frac{\beta}{\alpha}}$$

*Moreover, for  $\frac{d}{2} < \alpha$  we have*

$$\|f\|_{L^\infty(\mathbb{H}^d)} \lesssim \|f\|_{L^2(\mathbb{H}^d)}^{1-\frac{d}{2\alpha}} \|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^2(\mathbb{H}^d)}^{\frac{d}{2\alpha}}$$

**Lemma 2.5** (Boundedness of Riesz transform). *Let  $f \in C_0^\infty(\mathbb{H}^d)$ . Then for  $1 < p < \infty$  we have*

$$\|\nabla f\|_{L^p} \simeq \|(-\Delta)^{\frac{1}{2}} f\|_{L^p}$$

Lemma 2.4 can be proved using Littlewood-Paley projections as in Lemma 2.2. For a proof of Lemma 2.5, we refer to [66, Theorem 6.1].

**2.5.1. Norms for tensors and sections.** To extend the norms  $L^p(\mathbb{H}^d; \mathbb{R})$ ,  $W^{k,p}(\mathbb{H}^d; \mathbb{R})$  and  $H^k(\mathbb{H}^d; \mathbb{R})$  to vector-valued functions, we need to specify the pointwise norm for the vector. In the cases of interest, i.e., when the vector space is  $\mathbb{R}^n$  or  $\mathfrak{so}(n)$ , we use

$$|\varphi|_{\mathbb{R}^n}^2 = \langle \varphi, \varphi \rangle_{\mathbb{R}^n} := \sum_{k=1}^n (\varphi^k)^2, \quad |M|_{\mathfrak{so}(n)}^2 = \langle M, M \rangle_{\mathfrak{so}(n)} := \sum_{i,j=1}^n M_{ij}^2 \quad (2.35)$$

Observe that each norm arises from an inner product. When it is clear which one is being used from context, we will often omit the subscript  $\mathbb{R}^n$  and  $\mathfrak{so}(n)$ . Using these pointwise norms, we define the function norms  $L^p(\mathbb{H}^d; V)$ ,  $W^{k,p}(\mathbb{H}^d; V)$  and  $H^k(\mathbb{H}^d; V)$  ( $V = \mathbb{R}, \mathbb{R}^n$  or  $\mathfrak{so}(n)$ ) in the componentwise fashion. Functional inequalities in Lemma 2.1–Lemma 2.3 as well as those in Lemma 2.4 and Lemma 2.5 involving the fractional Laplacian still hold in this setting.

These norms may be further extended to  $V$ -valued tensors using the metric  $\mathbf{h}$ , in a similar fashion to (2.28). For instance, for a  $\mathbb{R}^n$ -valued covariant tensor  $\varphi_{a_1 \dots a_\ell}$ , we define these spaces using the pointwise norm

$$|\varphi| = \left( \mathbf{h}^{a_1 b_1} \dots \mathbf{h}^{a_\ell b_\ell} \langle \varphi_{a_1 \dots a_\ell}, \varphi_{b_1 \dots b_\ell} \rangle \right)^{\frac{1}{2}}$$

The basic functional inequalities stated in Lemma 2.1, Lemma 2.2 and Lemma 2.3 remain valid in the tensorial case, thanks to the following result.

**Lemma 2.6** (Diamagnetic inequality). *Let  $T$  be a smooth real-, vector- or matrix-valued tensor on  $\mathbb{H}^d$ . Then we have*

$$|\nabla |T|| \leq |\nabla T| \quad (2.36)$$

*in the distributional sense.*

More precisely, (2.36) holds when tested against smooth non-negative test functions. The proof proceeds by justifying the following formal computation:

$$|\nabla |T|| = \left| \nabla \sqrt{|T|^2} \right| = \left| \frac{\langle T, \nabla T \rangle}{|T|} \right| \leq |\nabla T|$$

It is clear that the key structural property we rely on is the fact that the pointwise norm arises from an inner product, which is parallel with respect to the covariant derivative  $\nabla$  (i.e.,  $\nabla \langle X, Y \rangle = \langle \nabla X, Y \rangle + \langle X, \nabla Y \rangle$ ). We omit the straightforward details for an actual proof.

Next, we consider objects defined on  $\mathcal{M} = \mathbb{R} \times \mathbb{H}^d$ , or more generally on its subsets of the form  $\mathcal{M}_I = I \times \mathbb{H}^d$ . Here the Lorentzian metric  $\boldsymbol{\eta}$  is unsuitable for defining pointwise norms for ( $V$ -valued) tensors, since it is not positive definite. To rectify this issue, we introduce a Riemannian metric  $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_{\alpha\beta})$  defined by  $\tilde{\boldsymbol{\eta}}_{00} = 1$ ,  $\tilde{\boldsymbol{\eta}}_{0a} = 0$  and  $\tilde{\boldsymbol{\eta}}_{ab} = \mathbf{h}_{ab}$ . Using this auxiliary metric  $\tilde{\boldsymbol{\eta}}$ , we define appropriate pointwise norms in the same fashion as above. For example, if  $\Psi = (\psi_\alpha)$  is a  $\mathbb{R}^n$ -valued 1-form on  $\mathcal{M}_I$ , then by the pointwise norm  $|\Psi|$  of  $\Psi$  we mean

$$|\Psi| = \left( \tilde{\boldsymbol{\eta}}^{\alpha\beta} \langle \psi_\alpha, \psi_\beta \rangle \right)^{\frac{1}{2}} = \left( \langle \psi_0, \psi_0 \rangle + \mathbf{h}^{ab} \langle \psi_a, \psi_b \rangle \right)^{\frac{1}{2}} \quad (2.37)$$

Since the metric  $\boldsymbol{\eta}$  is independent of  $t$  in the  $(t, x^a)$  coordinates on  $\mathcal{M}_I$ , the auxiliary metric  $\tilde{\boldsymbol{\eta}}$  is parallel with respect to the Levi-Civita connection. Hence the diamagnetic inequality (Lemma 2.6) holds for the above pointwise norms on  $\mathcal{M}_I$  as well, and Lemma 2.1, Lemma 2.2 and Lemma 2.3 extend to norms of tensors on  $\mathcal{M}_I$ .

2.5.2. *Norms for maps*  $\mathbb{H}^d \rightarrow \mathcal{N} \hookrightarrow \mathbb{R}^N$ . The main objects of study in the paper are smooth maps  $u : \mathbb{H}^d \rightarrow \mathcal{N}$  that tend to a fixed point  $u_\infty \in \mathcal{N}$  on the target at infinity, i.e.,

$$\mathbf{d}_{\mathcal{N}}(u(x), u_\infty) \rightarrow 0 \quad \text{as} \quad \mathbf{d}(x, \mathbf{0}) \rightarrow \infty \quad (2.38)$$

where  $\mathbf{d}$ , respectively  $\mathbf{d}_{\mathcal{N}}$ , is the geodesic distance function on  $\mathbb{H}^d$ , respectively  $\mathcal{N}$ . For technical purposes, it will be convenient to define a Sobolev norm for maps  $u : \mathbb{H}^d \rightarrow \mathcal{N}$  using the extrinsic representation. Assuming that  $\mathcal{N} \hookrightarrow \mathbb{R}^N$  isometrically and viewing  $u$  as an  $\mathbb{R}^N$ -valued function, (2.38) is equivalent to

$$|u(x) - u_\infty|_{\mathbb{R}^N} \rightarrow 0 \quad \text{as} \quad \mathbf{d}(x, \mathbf{0}) \rightarrow \infty$$

We will say that such a  $u$  has finite  $W^{s,p}$  norm with respect to  $u_\infty$  if

$$\|u - u_\infty\|_{W^{s,p}(\mathbb{H}^d; \mathbb{R}^N)} < \infty \quad (2.39)$$

where the norm  $W^{s,p}(\mathbb{H}^d; \mathbb{R}^N)$  for  $\mathbb{R}^N$ -valued functions is defined as above. We write

$$\|u\|_{W^{s,p}(\mathbb{H}^d; (\mathcal{N}, u_\infty))} := \|u - u_\infty\|_{W^{s,p}(\mathbb{H}^d; \mathbb{R}^N)} \quad (2.40)$$

The point of this notation is that one can make sense of the  $L^p$  spaces along with various functional inequalities relative to the fixed base point  $u_\infty$  on the target. In particular, appropriate versions of the inequalities in Lemma 2.1–Lemma 2.5 extend to smooth  $\mathcal{N}$ -valued maps which tend to  $u_\infty$  at infinity, as discussed above.

Given a  $u^*T\mathcal{N}$ -valued tensor  $\phi$ , which we view extrinsically as an  $\mathbb{R}^N$ -valued tensor, we also introduce the notation

$$\|\phi\|_{W^{s,p}(\mathbb{H}^d; T\mathcal{N})} := \|\phi\|_{W^{s,p}(\mathbb{H}^d; \mathbb{R}^N)} \quad (2.41)$$

For any positive  $s \in \mathbb{N}$ , we have

$$\|du\|_{W^{s-1,p}(\mathbb{H}^d; T\mathcal{N})} \simeq \|u\|_{W^{s,p}(\mathbb{H}^d; (\mathcal{N}, u_\infty))} \quad (2.42)$$

**2.6. The linear heat flow on  $\mathbb{H}^d$ .** Our analysis will make use of several heat flows on hyperbolic space. The most basic is the linear heat equation,

$$(\partial_s - \Delta)f = 0 \quad (2.43)$$

where, for the moment,  $f$  is a real-valued function on  $\mathbb{H}^d$ . Denote by  $p(x, y; s)$  the *heat kernel* on  $\mathbb{H}^d$ , that is, the kernel of the heat semi-group operator  $e^{s\Delta}$  acting on real valued functions on  $\mathbb{H}^d$ , i.e.,

$$e^{s\Delta}f(x) = \int_{\mathbb{H}^d} p(x, y; s)f(y) \, d\text{vol}_{\mathbf{h}} \quad (2.44)$$

Let  $J \subset \mathbb{R}^+$  be an interval extending from 0,  $f_0 \in L^2(\mathbb{H}^d)$  and  $G \in L^1(J; L^2(\mathbb{H}^d))$ . The unique solution in  $C([0, \infty); L^2(\mathbb{H}^d))$  to the inhomogeneous equation

$$\partial_s f - \Delta f = G, \quad f|_{s=0} = f_0 \quad (2.45)$$

is given by Duhamel's formula,

$$f(s) = e^{s\Delta}f_0 + \int_0^s e^{(s-s')\Delta}G(s') \, ds' \quad (2.46)$$

**2.6.1.  $L^2$  regularity theory.** The starting point for the linear heat equation (2.45) is the  $L^2$ -based regularity theory. We first sketch the energy integral approach for (2.45), which only relies on integration by parts. Based on this approach, we establish a more refined estimate for the homogeneous case (2.44) and present a simple extension to the inhomogeneous case (2.45), which will be used in Section 3 to establish well-posedness theory for the harmonic map heat flow.

Let  $f(s, \cdot)$  be a solution to (2.45), where we assume both  $f$  and  $G$  to be smooth and decaying sufficiently fast at the spatial infinity; the precise assumptions needed will be clear for each estimate we state below. Multiplying (2.45) by  $f$  and integrating by parts over  $\mathbb{H}^d$ , we obtain

$$\langle G | f \rangle = \langle \partial_s f | f \rangle + \langle -\Delta f | f \rangle = \frac{1}{2} \partial_s \|f\|_{L^2}^2 + s^{-1} \|s^{\frac{1}{2}} \nabla f\|_{L^2}^2$$

Integrating over the  $s$ -intervals  $(0, s)$  for  $s \in J$ , using Hölder's inequality and taking the square root, we arrive at

$$\|f\|_{L_{\frac{ds}{s}}^\infty(J; L^2)} + \|s^{\frac{1}{2}} \nabla f\|_{L_{\frac{ds}{s}}^2(J; L^2)} \lesssim \|f_0\|_{L^2} + \|G\|_{L_{\frac{ds}{s}}^1(J; L^2)} \quad (2.47)$$

This bound is useful from the point of view of Cauchy problem, since the right-hand side depends only on the data  $f_0$  and  $G$ .

On the other hand, multiplying (2.45) by  $s^\alpha (-\Delta)^\alpha f$  with an appropriate  $\alpha \neq 0$ , we obtain the following sharp  $L^2$  parabolic regularity bound.

**Lemma 2.7.** *Let  $J \subset \mathbb{R}^+$  be an interval extending from 0. Let  $f$  be a solution to (2.45) with  $f_0 \in L^2$  and*

$$s^{1+k} (-\Delta)^k G \in L_{\frac{ds}{s}}^2(J; L^2(\mathbb{H}^d)), \quad s^k (-\Delta)^k f \in L_{\frac{ds}{s}}^2(J; L^2(\mathbb{H}^d))$$

*for some  $k \geq 0$ . Then  $f$  obeys the bound*

$$\begin{aligned} & \|s^{k+\frac{1}{2}} (-\Delta)^{k+\frac{1}{2}} f\|_{L_{\frac{ds}{s}}^\infty(J; L^2)} + \|s^{k+1} (-\Delta)^{k+1} f\|_{L_{\frac{ds}{s}}^2(J; L^2)} \\ & \lesssim \|s^k (-\Delta)^k f\|_{L_{\frac{ds}{s}}^2(J; L^2)} + \|s^{k+1} (-\Delta)^k G\|_{L_{\frac{ds}{s}}^2(J; L^2)} \end{aligned} \quad (2.48)$$

*Proof.* Multiplying the equation (2.45) by  $s^{2k+1} (-\Delta)^{2k+1} f$ , we have

$$\langle \partial_s f | s^{2k+1} (-\Delta)^{2k+1} f \rangle + \langle -\Delta f | s^{2k+1} (-\Delta)^{2k+1} f \rangle = \langle G | s^{2k+1} (-\Delta)^{2k+1} f \rangle$$

Using functional calculus of the self-adjoint operator  $-\Delta$ , the previous identity can be rewritten as

$$\begin{aligned} & \frac{1}{2} \partial_s \|s^{k+\frac{1}{2}} (-\Delta)^{k+\frac{1}{2}} f\|_{L^2}^2 + s^{-1} \|s^{k+1} (-\Delta)^{k+1} f\|_{L^2}^2 \\ & = \left(k + \frac{1}{2}\right) s^{-1} \langle s^k (-\Delta)^k f | s^{k+1} (-\Delta)^{k+1} f \rangle \\ & \quad + s^{-1} \langle s^{k+1} (-\Delta)^k G | s^{k+1} (-\Delta)^{k+1} f \rangle \end{aligned}$$

We may write  $J = (0, s_0)$  for some  $s_0 > 0$ . Integrating over the  $s$ -interval  $(s', s_0)$  and taking  $s' \rightarrow 0$ , we obtain

$$\begin{aligned} & \frac{1}{2} \|s^{k+\frac{1}{2}} (-\Delta)^{k+\frac{1}{2}} f\|_{L_{\frac{ds}{s}}^\infty(J; L^2)}^2 + \|s^{k+1} (-\Delta)^{k+1} f\|_{L_{\frac{ds}{s}}^2(J; L^2)}^2 \\ & \leq \left| \int_J \left\langle \left(k + \frac{1}{2}\right) s^k (-\Delta)^k f + s^{k+1} (-\Delta)^k G | s^{k+1} (-\Delta)^{k+1} f \right\rangle \frac{ds}{s} \right| \end{aligned}$$

Applying Cauchy-Schwarz to the right-hand side and absorbing  $s^{k+1}(-\Delta)^{k+1}f$  on the left-hand side, the desired inequality (2.48) follows.  $\square$

We now return to the study of  $L^2$  theory of (2.45) from the point of view of the Cauchy problem. The starting point is a thorough understanding of the homogenous case, where  $G = 0$  and  $f(s) = e^{s\Delta}f_0$ . To simplify the notation, we will often drop the subscript 0 in  $f_0$ . Starting from (2.47) and applying (2.48) inductively for  $k = \frac{1}{2}, \frac{3}{2}, \dots$ , the following  $L^2_{\frac{ds}{s}}$  estimate holds.

**Lemma 2.8.** *Let  $f \in L^2(\mathbb{H}^d)$  and  $\alpha \geq 0$ . Then*

$$\|s^\alpha(-\Delta)^\alpha e^{s\Delta}f\|_{L^2(\mathbb{H}^d)} \lesssim_\alpha \|f\|_{L^2(\mathbb{H}^d)} \quad (2.49)$$

By the same argument, we obtain appropriately  $s$ -weighted  $L^2_{\frac{ds}{s}}$  estimates for the  $L^2(\mathbb{H}^d)$  norm of all derivatives  $e^{s\Delta}f$ , but not for the  $L^2$  norm of  $e^{s\Delta}f$  itself. To compensate for this in the sequel we will also require the following  $s$ -weighted  $L^2_{\frac{ds}{s}}$  estimate on the  $L^2$  norm of  $\alpha$  derivatives of  $e^{s\Delta}f$  for any  $\alpha > 0$ .

**Lemma 2.9.** *Let  $f \in L^2(\mathbb{H}^d)$  and  $\alpha > 0$ . Then,*

$$\left( \int_0^\infty \|s^\alpha(-\Delta)^\alpha e^{s\Delta}f\|_{L^2(\mathbb{H}^d)}^2 \frac{ds}{s} \right)^{1/2} \lesssim_\alpha \|f\|_{L^2(\mathbb{H}^d)} \quad (2.50)$$

*Proof.* Let  $Y = Y(\mathbb{R}^+ \times \mathbb{H}^d)$ , and  $Y^* = Y^*(\mathbb{R} \times \mathbb{H}^d)$  denote the Banach spaces defined by the norms

$$\|g\|_Y^2 := \int_0^\infty \|s^\alpha(-\Delta)^\alpha g(s)\|_{L^2}^2 \frac{ds}{s}, \quad \|F\|_{Y^*}^2 = \int_0^\infty \|s^{-\alpha}(-\Delta)^{-\alpha} F(s)\|_{L^2}^2 \frac{ds}{s}$$

Define  $Tf := e^{s\Delta}f$ . Our goal is to show that  $T : L^2(\mathbb{H}^d) \rightarrow Y(\mathbb{R}^+ \times \mathbb{H}^d)$  with bounded norm. By the  $TT^*$  method it suffices to prove the dual estimate

$$\|T^*F\|_{L^2(\mathbb{H}^d)} \lesssim \|F\|_{Y^*(\mathbb{R}^+ \times \mathbb{H}^d)} \quad (2.51)$$

where  $T^*F := \int_0^\infty e^{s\Delta}F(s) \frac{ds}{s}$ . To prove (2.51) we write  $G = s^{-\alpha}(-\Delta)^\alpha F$  and square the left-hand side above. We have

$$\begin{aligned} & \left\| \int_0^\infty e^{s\Delta}F(s) \frac{ds}{s} \right\|_{L^2}^2 \\ &= \int_0^\infty \int_0^\infty \left\langle s^\alpha e^{s\Delta}(-\Delta)^\alpha G(s) \mid (s')^\alpha e^{s'\Delta}(-\Delta)^\alpha G(s') \right\rangle_{L^2} \frac{ds}{s} \frac{ds'}{s'} \\ &= \int_0^\infty \int_0^\infty s^\alpha (s')^\alpha \left\langle G(s) \mid (-\Delta)^{2\alpha} e^{(s+s')\Delta} G(s') \right\rangle_{L^2} \frac{ds}{s} \frac{ds'}{s'} \\ &= 2 \int_0^\infty \int_s^\infty s^\alpha (s')^\alpha \left\langle G(s) \mid (-\Delta)^{2\alpha} e^{(s+s')\Delta} G(s') \right\rangle_{L^2} \frac{ds}{s} \frac{ds'}{s'} \end{aligned}$$

where the last line follows by symmetry in  $s, s'$ . Using Cauchy-Schwarz and the estimate (2.49) we can bound the last line above by

$$\begin{aligned}
& 2 \int_0^\infty \int_s^\infty s^\alpha (s')^\alpha \left\langle G(s) \mid (-\Delta)^{2\alpha} e^{(s+s')\Delta} G(s') \right\rangle_{L^2} \frac{ds}{s} \frac{ds'}{s'} \\
& \lesssim \int_0^\infty \int_s^\infty \frac{s^\alpha (s')^\alpha}{(s+s')^{2\alpha}} \|G(s)\|_{L^2} \|G(s')\|_{L^2} \frac{ds}{s} \frac{ds'}{s'} \\
& \lesssim \int_0^\infty \int_s^\infty \left(\frac{s}{s'}\right)^\alpha \|G(s)\|_{L^2} \|G(s')\|_{L^2} \frac{ds}{s} \frac{ds'}{s'} \\
& \lesssim \|G\|_{L^2_{\frac{ds}{s}} L^2}^2 \simeq \|s^{-\alpha} (-\Delta)^{-\alpha} F\|_{L^2_{\frac{ds}{s}} L^2}^2
\end{aligned}$$

where the last line follows from Schur's test. This completes the proof.  $\square$

The above homogeneous  $L^2$  estimates extend to the inhomogeneous case (2.45) via the following estimates for the Duhamel integral.

**Lemma 2.10.** *Let  $J \subset \mathbb{R}^+$  be an interval extending from 0, i.e.,  $J = (0, s_0]$  for some  $s_0 > 0$ . Let  $G$  satisfy  $sG \in L^1_{\frac{ds}{s}}(J; L^2(\mathbb{H}^d))$ . Then for  $0 \leq \alpha < 1$ , we have*

$$\|s^\alpha (-\Delta)^\alpha \int_0^s s' e^{(s-s')\Delta} G(s') \frac{ds'}{s'}\|_{L^\infty_{\frac{ds}{s}}(J; L^2(\mathbb{H}^d))} \lesssim_\alpha \|sG\|_{L^1_{\frac{ds}{s}} \cap L^\infty_{\frac{ds}{s}}(J; L^2(\mathbb{H}^d))}$$

For  $0 < \alpha < 1$ , we have

$$\|s^\alpha (-\Delta)^\alpha \int_0^s s' e^{(s-s')\Delta} G(s') \frac{ds'}{s'}\|_{L^2_{\frac{ds}{s}}(J; L^2(\mathbb{H}^d))} \lesssim_\alpha \|sG\|_{L^1_{\frac{ds}{s}} \cap L^2_{\frac{ds}{s}}(J; L^2(\mathbb{H}^d))}$$

Finally, corresponding to the case  $\alpha = 1$  we have

$$\begin{aligned}
& \|s \partial_s \int_0^s s' e^{(s-s')\Delta} G(s') \frac{ds'}{s'}\|_{L^2_{\frac{ds}{s}}(J; L^2(\mathbb{H}^d))} \\
& + \|s \Delta \int_0^s s' e^{(s-s')\Delta} G(s') \frac{ds'}{s'}\|_{L^2_{\frac{ds}{s}}(J; L^2(\mathbb{H}^d))} \lesssim \|sG\|_{L^1_{\frac{ds}{s}} \cap L^2_{\frac{ds}{s}}(J; L^2(\mathbb{H}^d))}
\end{aligned}$$

*Proof.* The case  $\alpha = 1$  follows by the energy integral method, or more precisely, by the combination of (2.47) and the following variant of (2.48) in the case  $k = 0$ :

$$\|s^{\frac{1}{2}} (-\Delta)^{\frac{1}{2}} f\|_{L^\infty_{\frac{ds}{s}}(J; L^2)} + \|s \Delta f\|_{L^2_{\frac{ds}{s}}(J; L^2)} \lesssim \|s^{\frac{1}{2}} (-\Delta)^{\frac{1}{2}} f\|_{L^2_{\frac{ds}{s}}(J; L^2)} + \|sG\|_{L^2_{\frac{ds}{s}}(J; L^2)}$$

We skip the proof, which is similar to that of (2.48). Henceforth, we fix either  $r = \infty$  and  $0 \leq \alpha < 1$ , or  $r = 2$  and  $0 < \alpha < 1$ . In the former case, the summation  $(\sum (\cdot)^r)^{1/r}$  is to be interpreted as a supremum as usual.

Extending  $G$  by zero outside  $J$ , we may assume that  $J = \mathbb{R}^+$ . For any  $k \in \mathbb{Z}$ , we split

$$G_k(s, x) = 1_{\{2^k < s \leq 2^{k+1}\}} G(s, x)$$

and accordingly, we define

$$H_k(s, x) = \int_0^s s' e^{(s-s')\Delta} G_k(s', x) \frac{ds'}{s'}$$

Then by the triangle inequality, we have

$$\begin{aligned}
\|s^\alpha(-\Delta)^\alpha \int_0^s s' e^{(s-s')\Delta} G(s') \frac{ds'}{s'}\|_{L_{\frac{ds}{s}}^r(J;L^2)} &= \left\| \sum_k s^\alpha(-\Delta)^\alpha H_k \right\|_{L_{\frac{ds}{s}}^r L^2} \\
&\leq \left\| \sum_k 1_{0 < s \leq 2^{k+2}} s^\alpha(-\Delta)^\alpha H_k \right\|_{L_{\frac{ds}{s}}^r L^2} \\
&\quad + \left\| \sum_k s^\alpha(-\Delta)^\alpha H_k \right\|_{L_{\frac{ds}{s}}^r((2^{k+2}, \infty); L^2)} \\
&=: I + II
\end{aligned}$$

We first treat the term  $I$ . Note that  $H_k(s) = 0$  for  $s \in (0, 2^k]$  by the support property of  $G_k$ ; therefore

$$I = \left\| \sum_k 1_{2^k < s \leq 2^{k+2}} s^\alpha(-\Delta)^\alpha H_k \right\|_{L_{\frac{ds}{s}}^r L^2}$$

Since the intervals  $(2^k, 2^{k+2}]$  have finite overlap, we may estimate

$$\begin{aligned}
&\left\| \sum_k 1_{2^k < s \leq 2^{k+2}} s^\alpha(-\Delta)^\alpha H_k \right\|_{L_{\frac{ds}{s}}^r L^2} \\
&\lesssim \left( \sum_k \|s^\alpha(-\Delta)^\alpha H_k\|_{L_{\frac{ds}{s}}^r((2^k, 2^{k+2}]; L^2)}^r \right)^{1/r} \\
&\lesssim \left( \sum_k \left\| \int_{2^k}^s s^\alpha(s-s')^{-\alpha} s' \|G_k(s')\|_{L^2} \frac{ds'}{s'} \right\|_{L_{\frac{ds}{s}}^r(2^k, 2^{k+2}]}^r \right)^{1/r}
\end{aligned}$$

By Schur's test, it follows that

$$\left( \sum_k \left\| \int_{2^k}^s s^\alpha(s-s')^{-\alpha} s' \|G_k(s')\|_{L^2} \frac{ds'}{s'} \right\|_{L_{\frac{ds}{s}}^r(2^k, 2^{k+2}]}^r \right)^{1/r} \lesssim \left( \sum_k \|s' G_k\|_{L_{\frac{ds'}{s'}}^r L^2}^r \right)^{1/r}$$

Since the supports of  $G_k$  in  $s$  are disjoint, the last term is bounded by  $\|sG\|_{L_{\frac{ds}{s}}^r(\mathbb{R}^+; L^2)}$ .

Hence we have proved

$$I = \left\| \sum_k 1_{2^k < s \leq 2^{k+2}} s^\alpha(-\Delta)^\alpha H_k \right\|_{L_{\frac{ds}{s}}^r L^2} \lesssim \|sG\|_{L_{\frac{ds}{s}}^r(\mathbb{R}^+; L^2)}$$

Next, we pass to the term  $II$ . For  $s > 2^{k+2}$ , we may write

$$H_k(s) = e^{(s-2^{k+1})\Delta} \int_{2^k}^{2^{k+1}} s' e^{(2^{k+1}-s')\Delta} G(s', x) \frac{ds'}{s'}$$

Observe also that  $(s - 2^{k+1}) \sim s$ . Thanks to the assumption that either  $\alpha \geq 0$  and  $r = \infty$  or  $\alpha > 0$  and  $r = 2$ , the homogeneous estimates (2.49) and (2.50) imply that

$$\begin{aligned}
\|s^\alpha(-\Delta)^\alpha H_k(s, x)\|_{L_{\frac{ds}{s}}^r((2^{k+2}, \infty); L^2)} &\lesssim \|(s - 2^{k+1})^\alpha(-\Delta)^\alpha H_k(s, x)\|_{L_{\frac{ds}{s}}^r((2^{k+2}, \infty); L^2)} \\
&\lesssim \left\| \int_{2^k}^{2^{k+1}} s' e^{(2^{k+1}-s')\Delta} G_k(s') \frac{ds'}{s'} \right\|_{L^2} \\
&\lesssim \|s' G_k\|_{L_{\frac{ds'}{s'}}^1 L^2}
\end{aligned}$$

Summing over  $k$ , we obtain

$$II = \sum_k \|s^\alpha (-\Delta)^\alpha H_k(s, x)\|_{L_{\frac{ds}{s}}^{r((2k+2), \infty); L^2}} \lesssim \sum_k \|s' G_k\|_{L_{\frac{ds'}{s'}}^1 L^2} \leq \|sG\|_{L_{\frac{ds}{s}}^1(\mathbb{R}^+; L^2)}$$

as desired.  $\square$

**2.6.2. Extension to vector-valued functions.** The  $L^2$  theory for the real-valued solutions to (2.45) extends naturally to vector-valued functions, which possibly tend to a nonzero constant  $u_\infty$  at the spatial infinity. More precisely, on an interval  $J \subset \mathbb{R}^+$  extending from 0, consider the Cauchy problem

$$\begin{cases} (\partial_s - \Delta)u = G \\ u|_{s=0} = u_0 \end{cases} \quad (2.52)$$

where  $u_0 - u_\infty \in L^2(\mathbb{H}^d; \mathbb{R}^N)$  for some  $u_\infty \in \mathbb{R}^N$  and  $G \in L^1(J; L^2(\mathbb{H}^d; \mathbb{R}^N))$ . Then there exists a unique solution  $u$  to (2.52) on  $J$  in the class

$$u(s, x) - u_\infty \in C(J \cup \{0\}; L^2(\mathbb{H}^d; \mathbb{R}^N)).$$

To see this, simply observe that  $(\partial_s - \Delta)u_\infty = 0$  and apply the usual uniqueness argument for each component of  $u - u_\infty$ . The unique solution is given by Duhamel's formula

$$u - u_\infty = e^{s\Delta}(u_0 - u_\infty) + \int_0^s e^{(s-s')\Delta} G(s') ds'$$

Note that the estimates in Lemma 2.8, Lemma 2.9 and Lemma 2.10 extend to the right-hand side of this formula in the natural componentwise manner.

**2.6.3.  $L^p$  regularity theory.** We would also need  $L^p$  boundedness and regularity theory of the heat semi-group  $e^{s\Delta}$ . We present a general argument which relies only on integration by parts.

**Lemma 2.11.** *Let  $1 < p < \infty$ . For any real-valued function  $f \in C_0^\infty(\mathbb{H}^d)$  and  $s > 0$ , the following bounds hold.*

$$\|e^{s\Delta} f\|_{L^p} \lesssim \|f\|_{L^p} \quad (2.53)$$

$$\|s^{\frac{1}{2}} \nabla e^{s\Delta} f\|_{L^p} \lesssim \|f\|_{L^p} \quad (2.54)$$

$$\|s(-\Delta)e^{s\Delta} f\|_{L^p} \lesssim \|f\|_{L^p} \quad (2.55)$$

*Proof.* To ease the notation, we henceforth denote by  $f(s)$  the solution to the homogeneous heat equation with  $f(0) \in C_0^\infty(\mathbb{H}^d)$ . In what follows, we will present a formal argument, which can be easily made precise using the  $L^2$  theory that we have already established.

We begin with some simple reductions. First, observe that once (2.53) and (2.55) are established, (2.54) follows easily from interpolation and Lemma 2.5. Second, by self-adjointness of  $e^{s\Delta}$  and  $s(-\Delta)e^{s\Delta}$  and duality, it suffices to prove (2.53) and (2.55) in the case  $2 \leq p < \infty$ . Finally, by interpolation and the  $L^2$  theory that we have already established, it is enough to consider the case when  $p$  is an even integer greater than 2.

Let  $p = 2n$  for some  $n \geq 2$  and let  $0 < \delta \ll 1$  be a small parameter to be chosen below. As  $f$  is a solution to the homogeneous heat equation, observe that  $|f|^2$  obeys the equation

$$\partial_s |f|^2 - \Delta |f|^2 + 2|\nabla f|^2 = 0$$



Since  $\Delta f$  solves the homogeneous heat equation as well,  $|\Delta f|^2$  obeys the same equation as above. Multiplying by  $s^2$ , we obtain

$$\partial_s(s^2 |\Delta f|^2) - \Delta(s^2 |\Delta f|^2) + 2s^2 |\nabla \Delta f|^2 - 2s |\Delta f|^2 = 0$$

Rewriting the last term on the left-hand side, we have

$$\partial_s(s^2 |\Delta f|^2) - \Delta(s^2 |\Delta f|^2) + 2s^2 |\nabla \Delta f|^2 + 2s(\nabla f \cdot \nabla \Delta f) - 2s \nabla \cdot (\nabla f \Delta f) = 0$$

We introduce an auxiliary function

$$\psi^2 = |f|^2 + \delta^2 s^2 |\Delta f|^2$$

We claim that if  $\delta > 0$  is sufficiently small, then

$$\sup_{s \in \mathbb{R}^+} \|\psi(s)\|_{L^p}^p \lesssim \|\psi(0)\|_{L^p}^p \quad (2.56)$$

Since  $\psi(0) = |f(0)|$ , the desired estimates (2.53) and (2.55) would then follow.

It now only remains to verify the claim (2.56). Adding up the equations for  $|f|^2$  and  $s^2 |\Delta f|^2$ , we see that  $\psi^2$  solves the equation

$$(\partial_s - \Delta)\psi^2 + F = 2\delta^2 s \nabla \cdot (\nabla f \Delta f)$$

where

$$F = 2 \left( |\nabla f|^2 + \delta^2 s^2 |\nabla \Delta f|^2 + \delta^2 (\nabla f \cdot s \nabla \Delta f) \right)$$

Multiplying the equation by  $\psi^{2n-2}$  and integrating by parts over  $\mathbb{H}^d$ , we obtain

$$\begin{aligned} \frac{1}{n} \partial_s \int_{\mathbb{H}^d} \psi^{2n} + (n-1) \int_{\mathbb{H}^d} |\nabla \psi^2|^2 \psi^{2n-4} + \int_{\mathbb{H}^d} F \psi^{2n-2} \\ = -2(n-1)\delta \int_{\mathbb{H}^d} \nabla f \cdot \nabla \psi^2 (\delta s \Delta f) \psi^{2n-4} \end{aligned}$$

Since  $0 < \delta \leq 1$ , observe that  $F$  obeys a  $\delta$ -independent lower bound  $F \geq |\nabla f|^2$ . Integrating over  $s$ -intervals of the form  $(0, s)$ , we arrive at

$$\begin{aligned} \sup_{s \in \mathbb{R}^+} \|\psi(s)\|_{L^p}^p + \int_0^\infty \int_{\mathbb{H}^d} |\nabla \psi^2|^2 \psi^{2n-4} ds + \int_0^\infty \int_{\mathbb{H}^d} |\nabla f|^2 \psi^{2n-2} ds \\ \lesssim \|\psi(0)\|_{L^p}^p + \delta \int_0^\infty \int_{\mathbb{H}^d} |\nabla f \cdot \nabla \psi^2 (\delta s \Delta f) \psi^{2n-4}| ds \end{aligned}$$

By Cauchy-Schwarz and the inequality  $|\delta s \Delta f| \leq |\psi|$ , the second term on the right-hand side can be bounded from above by

$$\begin{aligned} \delta \int_0^\infty \int_{\mathbb{H}^d} |\nabla f| |\psi| |\nabla \psi^2| \psi^{2n-4} ds \\ \lesssim \delta \left( \int_0^\infty \int_{\mathbb{H}^d} |\nabla \psi^2|^2 \psi^{2n-4} ds + \int_0^\infty \int_{\mathbb{H}^d} |\nabla f|^2 \psi^{2n-2} ds \right) \end{aligned}$$

Hence choosing  $\delta > 0$  sufficiently small, this term can be absorbed into the left-hand side. The desired claim (2.56) now follows.  $\square$

### 3. AN OVERVIEW OF THE PROOF OF THEOREM 1.1: WAVE MAPS IN THE CALORIC GAUGE

In this section we set the stage for the proof of Theorem 1.1 and construct the caloric gauge.

Our main objective over the next several sections will be to establish a priori estimates of the following form.

**Proposition 3.1** (A priori estimates). *Let  $0 \in I \subset \mathbb{R}$  be a time interval and suppose that  $u : I \times \mathbb{H}^4 \rightarrow \mathcal{N}$  is a smooth wave map on  $I$ . Then there exists  $\epsilon_1 > 0$  and a constant  $C_0$  independent of  $I$ ,  $u$ , so that if  $\epsilon \leq \epsilon_1$  and*

$$\|(du, \partial_t u)|_{t=0}\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} < \epsilon \quad (3.1)$$

then

$$\sup_{t \in I} \|(du, \partial_t u)(t)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \leq C_0 \epsilon. \quad (3.2)$$

*Remark 6.* Since the wave maps system is critical with respect to the norm (3.1), such an a priori estimate is not sufficient to continue the solution  $u$  past  $I$ . In fact, we will prove that a certain controlling norm, stronger than the norm on the left-hand side of (3.2), is bounded from above by  $C\epsilon$ ; see Theorem 4.1 below. For the sake of exposition, we defer the precise definition of the controlling norm until Section 4.

We will prove Proposition 3.1 by establishing the following bootstrap hypothesis: There exists an  $\epsilon_1 > 0$  small enough and a constant  $C_0 > 0$  with the following property. Given any time interval  $I \ni 0$ ,  $\epsilon \leq \epsilon_1$ , and smooth wave map  $u$  on  $I$  with initial data satisfying (3.1) then

$$\sup_{t \in I} \|(du, \partial_t u)(t)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \leq 2C_0 \epsilon \quad (3.3)$$

$$\implies \sup_{t \in I} \|(du, \partial_t u)(t)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \leq C_0 \epsilon. \quad (3.4)$$

To prove that (3.3) implies (3.4), we employ the derivative formulation of the wave maps equation and use the caloric gauge introduced by Tao. As discussed in Section 1.1, the construction of this gauge relies on a detailed understanding of the harmonic map heat flow. Therefore, our first order of business will be a small data global well-posedness theory for the harmonic map heat flow from  $\mathbb{H}^4 \rightarrow \mathcal{N} \subset \mathbb{R}^N$ , established in Section 3.1. Relying on this theory, we will show how to construct the caloric gauge in Section 3.2. After a brief summary of the dynamic variables and equations of the caloric gauge in Section 3.3, a more detailed outline of the strategy for proving (3.4) from (3.3) will be given in Section 4.

**3.1. The harmonic map heat flow.** In what follows, we will often use the shorthand  $L^p$  for  $L^p(\mathbb{H}^4; \mathbb{R}^N)$ .

A map  $u : \mathbb{R}^+ \times \mathbb{H}^4 \rightarrow \mathcal{N}$  satisfies the *harmonic map heat flow* if

$$\partial_s u = \mathbf{h}^{ab} D_a \partial_b u \quad (3.5)$$

In local coordinates on the target  $\mathcal{N}$  this becomes

$$\partial_s u^k = \Delta_{\mathbb{H}^4} u^k + \mathbf{h}^{ab} \Gamma_{ij}^k(u) \partial_a u^i \partial_b u^j \quad (3.6)$$

When the target is embedded the harmonic map heat flow can be expressed in coordinates as

$$\partial_s u^K - \Delta_{\mathbb{H}^4} u^K = \mathbf{h}^{ab} S_{IJ}^K(u) \partial_a u^I \partial_b u^J \quad (3.7)$$

where  $S = S_{IJ}^K$  is the second fundamental form of the embedding  $\mathcal{N} \hookrightarrow \mathbb{R}^N$ . This is equivalent to the intrinsic formulation.

The foundation for our construction of the caloric gauge is the following small data global regularity result for (3.7).

**Lemma 3.2** (Existence/uniqueness of the harmonic map heat flow). *Let  $u_0 : \mathbb{H}^4 \rightarrow \mathcal{N} \subset \mathbb{R}^N$  be smooth initial data for (3.7) such that*

$$u_0(x) \rightarrow u_\infty \in \mathcal{N} \text{ as } \mathbf{d}(x; \mathbf{0}) \rightarrow \infty. \quad (3.8)$$

*Then, there exists  $\delta_1 > 0$  with the following property: If*

$$\|du_0\|_{H^1(\mathbb{H}^4; T\mathcal{N})} = \delta < \delta_1 \quad (3.9)$$

*then there exists a smooth global solution  $u(s, x)$  to the harmonic map heat flow equation (3.7) with initial data  $u_0$ , and satisfying the uniform estimate*

$$\begin{aligned} \|\Delta u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} + \sum_{\alpha \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}} \|s^\alpha (-\Delta)^{\alpha+1} u\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \\ + \|s(-\Delta)^2 u\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \lesssim \|du\|_{s=0} \|_{H^1(\mathbb{H}^4, T\mathcal{N})} \end{aligned} \quad (3.10)$$

*The solution  $u$  is unique in the class of maps  $u \in C([0, \infty); H^2(\mathbb{H}^4; (\mathcal{N}, u_\infty)))$  such that the left-hand side of (3.10) is finite. Moreover, we have the following uniform estimates on higher  $s$ -weighted derivatives of  $u$ ,*

$$\|s^\alpha (-\Delta)^{\alpha+1} u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} \lesssim_\alpha \|du\|_{s=0} \|_{H^1(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq 0 \quad (3.11)$$

$$\|s^\alpha (-\Delta)^{\alpha+1} u\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \lesssim_\alpha \|du\|_{s=0} \|_{H^1(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha > 0 \quad (3.12)$$

$$\|s^\alpha (-\Delta)^\alpha \partial_s u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} \lesssim_\alpha \|du\|_{s=0} \|_{H^1(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq 0 \quad (3.13)$$

$$\|s^\alpha (-\Delta)^\alpha \partial_s u\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \lesssim_\alpha \|du\|_{s=0} \|_{H^1(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq \frac{1}{2} \quad (3.14)$$

*Proof of Lemma 3.2.* We work in the extrinsic setting with (3.7). To ease the notation, we will use lower case alphabets (such as  $i, j, k$ ) instead of capital letters to denote the indices  $1, \dots, N$ .

**Step 0.** The proof follows by the usual Picard iteration scheme which is conducted in a bounded subset of the space

$$\begin{aligned} \mathcal{X}_0(\mathbb{R}_+) := \{u : u - u_\infty \in C^0(\mathbb{R}^+; H^2(\mathbb{H}^4; \mathbb{R}^N)) : \|u\|_{\mathcal{X}_0(\mathbb{R}^+)} < \infty, \\ u(0, \cdot) = u_0 \in H^2(\mathbb{H}^4; (\mathcal{N}, u_\infty))\} \end{aligned} \quad (3.15)$$

where the norm  $\|\cdot\|_{\mathcal{X}_0(J)}$  for any  $s$ -interval  $J \subset \mathbb{R}^+$  is defined as

$$\begin{aligned} \|u\|_{\mathcal{X}_0(J)} := \|\Delta u\|_{L_{\frac{ds}{s}}^\infty(J; L^2)} + \|s(-\Delta)^2 u\|_{L_{\frac{ds}{s}}^2(J; L^2)} \\ + \sum_{\alpha \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}} \|s^\alpha (-\Delta)^{\alpha+1} u\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(J; L^2)} \end{aligned} \quad (3.16)$$

The techniques in the proof are standard and we give a rough sketch introducing the main ideas by establishing the a priori estimates (3.10) in the next step. Before proceeding with the a priori estimates we clear up a minor point, namely that a solution  $u \in \mathcal{X}_0(\mathbb{R}^+)$  to (3.7) that we construct actually lands in  $\mathcal{N} \subset \mathbb{R}^N$  for all  $s \in \mathbb{R}^+$ , that is,

$$u(s, x) \in \mathcal{N} \quad \forall (s, x) \in [0, \infty) \times \mathbb{H}^4 \quad (3.17)$$

We argue as in [45, Section 5.2]. Let  $u : \mathbb{R}^+ \times \mathbb{H}^4 \rightarrow \mathbb{R}^N$  be a solution to (3.7) with  $u \in \mathcal{X}_0(\mathbb{R}^+)$ . Let  $\mathcal{N}_\epsilon$  be an  $\epsilon$ -neighborhood of  $\mathcal{N}$  and let

$$\pi_{\mathcal{N}} : \mathcal{N}_\epsilon \rightarrow \mathcal{N} \quad (3.18)$$

be the smooth, closest point projection map, i.e., for  $w \in \mathcal{N}_\epsilon$  we have

$$|\pi_{\mathcal{N}}(w) - w| = \inf_{u \in \mathcal{N}} |u - w| \quad (3.19)$$

Note that for  $w \in \mathcal{N}$ ,  $\nabla \pi_{\mathcal{N}}(w) : \mathbb{R}^N \rightarrow T_w \mathcal{N}$  is an orthogonal projection, and moreover,  $\nabla^2 \pi_{\mathcal{N}}(w) : T_w \mathcal{N} \otimes T_w \mathcal{N} \rightarrow (T_w \mathcal{N})^\perp$  is precisely the second fundamental form, i.e.,  $\nabla^2 \pi_{\mathcal{N}}(w)(\cdot, \cdot) = -S(w)(\cdot, \cdot)$ .

Now, set  $\rho(u) := |\pi_{\mathcal{N}}(u) - u|^2$ , and note that  $\rho(u(0, \cdot)) = 0$ . We have

$$\partial_s \rho(u) - \Delta \rho(u) = -2 |\nabla(\pi_{\mathcal{N}}(u) - u)|^2 - 2 \sum_{j=1}^N (\pi_{\mathcal{N}}^j(u) - u^j) (\partial_s - \Delta)(\pi_{\mathcal{N}}^j(u) - u^j)$$

We claim that the last term on the right above is  $\equiv 0$ . Assuming this last fact, we see that  $\partial_s \rho(u) - \Delta \rho(u) \leq 0$  and an application of the maximum principle concludes the proof. To prove the last term is  $\equiv 0$  we compute

$$\begin{aligned} (\partial_s - \Delta)(\pi_{\mathcal{N}}^j(u) - u^j) &= \partial_\ell \pi_{\mathcal{N}}^j(u) (\partial_s u^\ell - \Delta u^\ell) - \partial_k \partial_\ell \pi_{\mathcal{N}}^j(u) \partial^a u^k \partial_a u^\ell \\ &\quad - S_{\ell k}^j(u) \partial^a u^k \partial_a u^\ell \\ &= \partial_\ell \pi_{\mathcal{N}}^j(u) S^\ell(u)(du, du) + S_{\ell k}^j(u) \partial^a u^k \partial_a u^\ell - S_{\ell k}^j(u) \partial^a u^k \partial_a u^\ell \\ &= 0 \end{aligned}$$

where above we have used the fact that  $S(u)(du, du) \in (T_u \mathcal{N})^\perp$  to show that the first term in the third line above vanishes.

**Step 1: A priori estimates.** We now prove the a priori estimates (3.10). Let  $u(s, x)$  be a smooth solution to (3.7) on an  $s$ -interval  $I \subset \mathbb{R}^+$ . Define  $v(s, x)$  by

$$v := \Delta(u - u_\infty) = \Delta u \quad (3.20)$$

where we view  $v(s, x)$  as a vector in  $\mathbb{R}^N$ . Then  $v$  satisfies the heat equation

$$\partial_s v - \Delta v = G, \quad (3.21)$$

where  $G(s, x) \in \mathbb{R}^N$  is the vector with components

$$G^k := \Delta(S_{j\ell}^k(u) \nabla^a u^j \nabla_a u^\ell) \quad (3.22)$$

Now, given an  $s$ -interval  $J \subset \mathbb{R}^+$ , we define the norms  $X(J)$  and  $X_0(J)$  by

$$\begin{aligned} \|v\|_{X_0(J)} &:= \|v\|_{L_{\frac{ds}{s}}^\infty(J; L^2)} + \|s \Delta v\|_{L_{\frac{ds}{s}}^2(J; L^2)} \\ &\quad + \sum_{\alpha \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}} \|s^\alpha (-\Delta)^\alpha v\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(J; L^2)} \end{aligned} \quad (3.23)$$

Note that  $\|v\|_{X_0(J)} = \|u\|_{X_0(J)}$ .

By Duhamel's formula, the solution  $v$  to (3.21) admits the (componentwise) representation

$$v(s) = e^{s\Delta}v_0 + \int_0^s s' e^{(s-s')\Delta} G(s') \frac{ds'}{s'} \quad (3.24)$$

where  $v_0 = \Delta u_0 = \Delta(u_0 - u_\infty)$ . Let  $J \subset \mathbb{R}^+$  be any  $s$ -interval extending from 0. By Lemma 2.8 and Lemma 2.9, we have

$$\|e^{s\Delta}v_0\|_{X_0(\mathbb{R}^+)} \lesssim \|v_0\|_{L^2} \simeq \|du_0\|_{H^1(\mathbb{H}^4; T\mathcal{N})} \quad (3.25)$$

On the other hand, by Lemma 2.10 it follows that

$$\left\| \int_0^s s' e^{(s-s')\Delta} G(s') \frac{ds'}{s'} \right\|_{X_0(J)} \lesssim \|sG\|_{L^1_{\frac{ds}{s}} \cap L^\infty_{\frac{ds}{s}}(J; L^2)} \quad (3.26)$$

For each fixed  $s \in \mathbb{R}^+$ , we claim that the following bound holds for  $G$ :

$$\|sG\|_{L^1_{\frac{ds}{s}} \cap L^\infty_{\frac{ds}{s}}(J; L^2)} \lesssim (1 + \|v\|_{X_0(J)} + \|v\|_{X_0(J)}^2) \|v\|_{X_0(J)}^2 \quad (3.27)$$

Assuming this bound holds, a usual continuity argument with  $\delta_1$  sufficiently small leads to the *a priori* estimate

$$\|v\|_{X_0(\mathbb{R}^+)} \lesssim \|v(0)\|_{L^2} \quad (3.28)$$

which implies (3.10).

We now turn to the proof of (3.27). From the definition of the norm  $X_0(J)$ , it is clear that (3.27) would follow once we establish

$$\|G\|_{L^2} \lesssim (1 + \|v\|_{L^2} + \|v\|_{L^2}^2) \|(-\Delta)^{\frac{1}{2}} v\|_{L^2}^2 + \|(-\Delta)^{\frac{1}{4}} v\|_{L^2} \|(-\Delta)^{\frac{3}{4}} v\|_{L^2} \quad (3.29)$$

for every fixed  $s \in \mathbb{R}^+$ . We begin by expanding  $G$  as defined in (3.22).

$$\begin{aligned} G^k &= \mathbf{h}^{cd} \nabla_c \nabla_d S_{ij}^k(u) \mathbf{h}^{ab} \nabla_a u^i \nabla_b u^j \\ &= \partial_{m\ell}^2 S_{ij}^k \mathbf{h}^{cd} \mathbf{h}^{ab} \nabla_c u^m \nabla_d u^\ell \nabla_a u^i \nabla_b u^j \\ &\quad + \partial_\ell S_{ij}^k(u) \Delta u^\ell \mathbf{h}^{ab} \nabla_a u^i \nabla_b u^j + 4 \partial_\ell S_{ij}^k(u) \mathbf{h}^{cd} \mathbf{h}^{ab} \nabla_d u^\ell \nabla_c \nabla_a u^i \nabla_b u^j \\ &\quad + 2 S_{ij}^k(u) \mathbf{h}^{cd} \mathbf{h}^{ab} \nabla_c \nabla_a u^i \nabla_d \nabla_b u^j + 2 S_{ij}^k(u) \mathbf{h}^{cd} \mathbf{h}^{ab} \nabla_c \nabla_d \nabla_a u^i \nabla_b u^j \end{aligned}$$

We express the above schematically as

$$\begin{aligned} G &= \partial^{(2)} S(u) (\nabla u, \nabla u, \nabla u, \nabla u) + \partial S(u) (\Delta u, \nabla u, \nabla u) + \partial S(u) (\nabla u, \nabla^2 u, \nabla u) \\ &\quad + S(u) (\nabla^2 u, \nabla^2 u) + S(u) (\nabla^3 u, \nabla u) \end{aligned}$$

We now take the  $L^2$  norm and estimate each term as follows. Below we will often use the assumption that the second fundamental form  $S$  and its derivatives are bounded, i.e.,  $|S(u)| + |\partial S(u)| + |\partial^{(2)} S(u)| \leq C$ . Using the Gagliardo-Nirenberg and Sobolev inequalities, we estimate

$$\|\partial^{(2)} S(u) (\nabla u, \nabla u, \nabla u, \nabla u)\|_{L^2} \lesssim \|\nabla u\|_{L^8}^4 \lesssim \|\nabla u\|_{L^4}^2 \|\Delta u\|_{L^4}^2 \lesssim \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2$$

Similarly,

$$\|\partial S(u) (\Delta u, \nabla u, \nabla u)\|_{L^2} \lesssim \|\Delta u\|_{L^4} \|\nabla u\|_{L^8}^2 \lesssim \|\Delta u\|_{L^4}^2 \|\nabla u\|_{L^4}^2 \lesssim \|v\|_{L^2} \|\nabla v\|_{L^2}^2$$

Exactly the same argument applies to  $\partial S(u)(\nabla u, \nabla^2 u, \nabla u)$ . By the Sobolev inequality, it also follows that

$$\begin{aligned} \|S(u)(\nabla^2 u, \nabla^2 u)\|_{L^2} &\lesssim \|\Delta u\|_{L^4}^2 \lesssim \|\nabla v\|_{L^2}^2 \\ \|S(u)(\nabla^3 u, \nabla u)\|_{L^2} &\lesssim \|\nabla^3 u\|_{L^{\frac{8}{3}}} \|\nabla u\|_{L^8} \lesssim \|(-\Delta)^{\frac{3}{4}} v\|_{L^2} \|(-\Delta)^{\frac{1}{4}} v\|_{L^2} \end{aligned}$$

These estimates imply (3.29) as desired.

**Step 2: Existence and uniqueness.** To establish existence of solutions one can now set up the usual iteration argument in the space

$$\mathcal{X}_{0,\delta} := \{u \in \mathcal{X}_0(\mathbb{R}^+) : \|u\|_{\mathcal{X}_0(\mathbb{R}^+)} \leq C\delta\} \quad (3.30)$$

Define a sequence of maps by setting  $u_{-1}(s, x) := u_\infty$  and  $u_j$  for  $j \geq 0$  to be the solution to the linear heat equation

$$\begin{aligned} (\partial_s - \Delta)u_j &= S(u_{j-1})(du_{j-1}, du_{j-1}) \\ u_j(0, x) &= u_0 \end{aligned} \quad (3.31)$$

One now shows using the same ideas used to prove (3.10) that the sequence is well defined in  $\mathcal{X}_{0,\delta}$  and Cauchy in  $\mathcal{X}_{0,\delta}$  for  $\delta$  small enough. The uniqueness of solutions to (3.7) in the space  $\mathcal{X}_0(\mathbb{R}^+)$ , as well as smoothness of  $u$  on  $\mathbb{R}^+ \times \mathbb{H}^4$  (given that  $u_0$  is smooth) follow from a standard argument which we do not carry out here.

**Step 3: Smoothing bounds.** Next, we turn to the bounds (3.11) and (3.12). The case  $\alpha < 1$  follows from Lemma 2.8, Lemma 2.9, Lemma 2.10 and (3.27). To treat higher derivatives, we apply Lemma 2.7 with  $k \geq 1$ . We sketch the case  $k = 1$  and leave the general case  $k \geq 1$  to the reader.

Let  $J \subset \mathbb{R}^+$  be any interval extending from 0. Applying Lemma 2.7 with  $k = 1$  to (3.21), it follows that

$$\|s^{\frac{3}{2}}(-\Delta)^{\frac{3}{2}}v\|_{L_{\frac{ds}{s}}^\infty(J;L^2)} + \|s^2(-\Delta)^2v\|_{L_{\frac{ds}{s}}^2(J;L^2)} \lesssim \|s\Delta v\|_{L_{\frac{ds}{s}}^2(J;L^2)} + \|s\Delta G\|_{L_{\frac{ds}{s}}^2(J;L^2)}$$

The first term on the right-hand side is bounded by  $\|du_0\|_{H^1(\mathbb{H}^4;T\mathcal{N})}$  thanks to Step 1. Proceeding as in the proof of (3.27) and using interpolation, it can be checked that

$$\|s\Delta G\|_{L_{\frac{ds}{s}}^2(J;L^2)} \lesssim \left(\|v\|_{X_0(J)} + \|v\|_{X_0(J)}^5\right) \sum_{\alpha \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}} \|s^{1+\alpha}(-\Delta)^{1+\alpha}v\|_{L_{\frac{ds}{s}}^2(J;L^2)}$$

By interpolation and Young's inequality, for every  $\tilde{\delta} > 0$  there exists  $C_{\tilde{\delta}} > 0$  such that

$$\sum_{\alpha \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}} \|s^{1+\alpha}(-\Delta)^{1+\alpha}v\|_{L_{\frac{ds}{s}}^2(J;L^2)} \leq \tilde{\delta} \|s^2(-\Delta)^2v\|_{L_{\frac{ds}{s}}^2(J;L^2)} + C_{\tilde{\delta}} \|v\|_{X_0(J)}$$

Recall that  $\|v\|_{X_0(J)} \lesssim \|du_0\|_{H^1(\mathbb{H}^4;T\mathcal{N})}$  by Step 1. Choosing  $\tilde{\delta} > 0$  sufficiently small depending on  $\|du_0\|_{H^1(\mathbb{H}^4;T\mathcal{N})}$ , we may absorb the first term to the left-hand side, provided it is finite. By a standard continuity argument in  $s$  (which ensures the required finiteness), we conclude that

$$\|s^{\frac{3}{2}}(-\Delta)^{\frac{3}{2}}v\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+;L^2)} + \|s^2(-\Delta)^2v\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+;L^2)} \lesssim \|du_0\|_{H^1(\mathbb{H}^4;T\mathcal{N})}$$

**Step 4: Regularity in  $s$ .** Finally, we establish (3.13) and (3.14). The idea is simply to use the harmonic map heat flow equation

$$\partial_s u - \Delta u = \tilde{G}$$

where  $\tilde{G}$  is defined as

$$\tilde{G}^k = S_{ij}^k(u) \nabla^a u^i \nabla_a u^j$$

Let  $r = 2$  or  $\infty$ . Applying  $s^\alpha (-\Delta)^\alpha$  to the equation and using  $v = \Delta v$ , we derive

$$\|s^\alpha (-\Delta)^\alpha \partial_s u\|_{L_{\frac{ds}{s}}^r(\mathbb{R}^+; L^2)} \leq \|s^\alpha (-\Delta)^\alpha v\|_{L_{\frac{ds}{s}}^r(\mathbb{R}^+; L^2)} + \|s^\alpha (-\Delta)^\alpha \tilde{G}\|_{L_{\frac{ds}{s}}^r(\mathbb{R}^+; L^2)}$$

The first term on the right-hand side is acceptable; indeed, by the previous step,

$$\|s^\alpha (-\Delta)^\alpha v\|_{L_{\frac{ds}{s}}^r(\mathbb{R}^+; L^2)} \lesssim_\alpha \|du_0\|_{H^1(\mathbb{H}^4, T\mathcal{N})}$$

for every  $\alpha \geq 0$  and  $r = \infty$ , or  $\alpha > 0$  and  $r = 2$ . On the other hand, the second term can be bounded by  $C_\alpha \|du_0\|_{H^1(\mathbb{H}^4, T\mathcal{N})}^2$  for  $\alpha = 2, 3, \dots$ , by using the Leibniz rule to expand  $s^\alpha (-\Delta)^\alpha \tilde{G} = s^\alpha (-\Delta)^{\alpha-1} G$  and applying the bounds (3.11) and (3.12). By interpolation, it remains to handle the cases with the least number of derivatives. Using Gagliardo-Nirenberg and Sobolev, we have

$$\begin{aligned} \|\tilde{G}\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} &\lesssim \|\nabla u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^4)}^2 \lesssim \|v\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)}^2 \\ \|s^{\frac{1}{2}} (-\Delta)^{\frac{1}{2}} \tilde{G}\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} &\lesssim \|s^{\frac{1}{2}} \nabla \tilde{G}\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \\ &\lesssim \|\nabla u\|_{L_{\frac{ds}{s}}^6(\mathbb{R}^+)}^3 + \|\nabla u\|_{L^4} \|\nabla^2 u\|_{L^4} \|v\|_{L_{\frac{ds}{s}}^2} \\ &\lesssim \left( \|v\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} + \|v\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)}^2 \right) \|(-\Delta)^{\frac{1}{2}} v\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)}, \end{aligned}$$

both of which are acceptable.  $\square$

As is usual for proofs using Picard iteration, the solution map  $u_0 \mapsto u$  is smooth with respect to the norms in the estimates (3.11)–(3.14). We record precise bounds in the following statement.

**Lemma 3.3** (Linearized harmonic map heat flow). *Let  $I \subset \mathbb{R}$  be an interval, and let  $u_0 = (u_{0,t})_{t \in I}$  be a smooth 1-parameter family of maps  $\mathbb{H}^4 \rightarrow \mathcal{N} \subset \mathbb{R}^N$ . There exists  $\delta_2 > 0$  independent of  $t \in I$  and with  $\delta_2 < \delta_1$  where  $\delta_1$  is the small constant in Lemma 3.2, so that the following holds: Suppose that*

$$u_{0,t}(x) \rightarrow u_\infty \in \mathcal{N} \text{ as } \mathbf{d}(x; \mathbf{0}) \rightarrow \infty, \quad \|du_{0,t}\|_{H^1(\mathbb{H}^4; T\mathcal{N})} = \delta < \delta_2$$

*for every  $t \in I$ , and let  $u = u_t(s, x)$  be the global solution to (3.5) with initial data  $u_{0,t}$ . Then  $u$  is smooth in  $(t, s, x) \in I \times \mathbb{R}^+ \times \mathbb{H}^4$  and obeys the following bounds for each  $t \in I$ :*

$$\|s^\alpha (-\Delta)^{\alpha+\frac{1}{2}} \partial_t u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} \lesssim_{\delta, \alpha} \|\partial_t u|_{s=0}\|_{H^1(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq 0 \quad (3.32)$$

$$\|s^\alpha (-\Delta)^{\alpha+\frac{1}{2}} \partial_t u\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \lesssim_{\delta, \alpha} \|\partial_t u|_{s=0}\|_{H^1(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha > 0 \quad (3.33)$$

$$\|s^\alpha (-\Delta)^{\alpha-\frac{1}{2}} \partial_t \partial_s u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} \lesssim_{\delta, \alpha} \|\partial_t u|_{s=0}\|_{H^1(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq 0 \quad (3.34)$$

$$\|s^\alpha (-\Delta)^{\alpha-\frac{1}{2}} \partial_t \partial_s u\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \lesssim_{\delta, \alpha} \|\partial_t u|_{s=0}\|_{H^1(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq \frac{1}{2} \quad (3.35)$$

*Remark 7.* For technical convenience we have introduced a new small constant  $\delta_2 > 0$  in the statement of Lemma 3.3, which is acceptable thanks to the small data assumption. We remark that this can be avoided by more carefully choosing a function space for the harmonic maps heat flow  $u$  which is divisible in  $s \in \mathbb{R}^+$ . Here divisibility of a space  $\mathcal{Y}(J)$  refers to the fact that for any function  $u \in \mathcal{Y}(J)$  there exists a partition of  $J$  into a controlled number of subintervals  $\{J_k\}$  on each of which the  $\mathcal{Y}(J_k)$  norm of  $u$  is small.

Let  $v' = \partial_t u$ . Then  $v'$  obeys the equation

$$(\partial_s - \Delta)v' = G' \quad (3.36)$$

where  $G'$  is given by

$$\begin{aligned} (G')^k &= \partial_t(S_{ij}^k(u)\nabla^a u^i \nabla_a u^j) \\ &= \partial_t S_{ij}^k(u)(v')^i \nabla^a u^j + 2S_{ij}^k(u)\nabla^a (v')^i \nabla_a u^j \end{aligned}$$

Hence  $v'$  solves a *linearized harmonic map heat flow* about  $u$ . Lemma 3.3 is proved by analyzing this flow.

*Proof of Lemma 3.3.* Fix  $t \in I$ . We begin by showing that for  $\delta < \delta_2$  where  $\delta_2$  is independent of  $t$  and  $I$  the following analogue of (3.10) holds:

$$\|(-\Delta)^{\frac{1}{2}}v'\|_{X_0(\mathbb{R}^+)} \lesssim_\delta \|\partial_t u_0\|_{H^1(\mathbb{H}^4; T\mathcal{N})} \quad (3.37)$$

Indeed, applying Duhamel's principle to (3.36),

$$v'(s) = e^{s\Delta}v'(0) + \int_0^s s' e^{(s-s')\Delta} G'(s') \frac{ds'}{s'}$$

Let  $J \subseteq \mathbb{R}^+$  be any fixed  $s$ -interval. By Lemma 2.8 and Lemma 2.9, we have

$$\|(-\Delta)^{\frac{1}{2}}e^{s\Delta}v'(0)\|_{X_0(J)} \lesssim \|(-\Delta)^{\frac{1}{2}}v'(0)\|_{L^2} \simeq \|\partial_t u_0\|_{H^1(\mathbb{H}^4; T\mathcal{N})}$$

which is acceptable. On the other hand, by Lemma 2.10, we have

$$\|(-\Delta)^{\frac{1}{2}} \int_0^s s' e^{(s-s')\Delta} G'(s') \frac{ds'}{s'}\|_{X_0(J)} \lesssim \|\nabla G'\|_{L^1_{\frac{ds}{s}} \cap L^2_{\frac{ds}{s}}(J; L^2)}$$

We claim that

$$\|\nabla G'\|_{L^1_{\frac{ds}{s}} \cap L^\infty_{\frac{ds}{s}}(J; L^2)} \lesssim (1 + \|u\|_{X_0(J)}^2) \|u\|_{X_0(J)} \|(-\Delta)^{\frac{1}{2}}v'\|_{X_0(J)} \quad (3.38)$$

Note the linear factor  $\|(-\Delta)^{\frac{1}{2}}v'\|_{X_0(J)}$ . Choosing  $\delta_2 < \delta_1$  we have  $\|u\|_{X_0(J)} \lesssim \delta$  by Lemma 3.2. Choosing  $\delta_2$  smaller if necessary, we may prove

$$\|(-\Delta)^{\frac{1}{2}}v'\|_{X_0(J)} \leq C \|\partial_t u_0\|_{H^1(\mathbb{H}^4; T\mathcal{N})} + \frac{1}{2} \|(-\Delta)^{\frac{1}{2}}v'\|_{X_0(J)}$$

Note that the last term can be absorbed into the left-hand side which proves the desired a priori estimate (3.37).

The estimate (3.38) follows by expanding (schematically)

$$\begin{aligned} \nabla G' &= \partial^{(2)} S(u)(v', \nabla u, \nabla u, \nabla u) + \partial S(u)(\nabla v', \nabla u, \nabla u) + \partial S(u)(v', \nabla^2 u, \nabla u) \\ &\quad + S(u)(\nabla^2 v', \nabla u) + S(u)(\nabla v', \nabla^2 u) \end{aligned}$$

and treating each term using Gagliardo-Nirenberg and Sobolev, in the fashion of Step 1 in the proof of Lemma 3.2. We leave the routine details to the reader.

We now sketch the rest of the proof. Once the key a priori estimate (3.37) is proved, the higher regularity bounds (3.32) and (3.33) can be established using



Lemma 2.7 as in the proof of (3.11) and (3.12). Moreover, the bounds (3.34) and (3.35) for  $\partial_s \partial_t u = \partial_s v'$  follow from the equation (3.36) as in the proof of (3.13) and (3.14).  $\square$

Proceeding as in Lemma 3.3 but commuting the harmonic map heat flow equation with  $-\Delta$ , we obtain a priori estimates for  $u$  with one less power of  $s$  in terms of the  $H^3$  norm of  $du|_{s=0}$ . Considering a smooth one parameter family of maps  $u_t$  and taking  $\partial_t$ , we also get similar estimates for  $\partial_t u$  in terms of the  $H^3$  norm of  $\partial_t u|_{s=0}$ . We record these bounds in the following lemma. Such bounds will be useful later in the context of upgrading a priori estimates to well-posedness of the wave map equation.

**Lemma 3.4** (A priori estimate for higher derivatives). *Let  $u_0$  be a smooth map  $\mathbb{H}^4 \rightarrow \mathcal{N}$  satisfying the hypothesis of Lemma 3.2. Assume furthermore that*

$$\|du_0\|_{H^3(\mathbb{H}^4; T\mathcal{N})} < \infty$$

*Then the solution  $u$  to the harmonic map heat flow with initial data  $u_0$  given by Lemma 3.2 obeys the following estimates.*

$$\begin{aligned} \|s^{\alpha-1}(-\Delta)^{\alpha+1}u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} &\lesssim_{\delta, \alpha} \|du|_{s=0}\|_{H^3(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq 1 \\ \|s^{\alpha-1}(-\Delta)^{\alpha+1}u\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} &\lesssim_{\delta, \alpha} \|du|_{s=0}\|_{H^3(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha > 1 \\ \|s^{\alpha-1}(-\Delta)^\alpha \partial_s u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} &\lesssim_{\delta, \alpha} \|du|_{s=0}\|_{H^3(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq 1 \\ \|s^{\alpha-1}(-\Delta)^\alpha \partial_s u\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} &\lesssim_{\delta, \alpha} \|du|_{s=0}\|_{H^3(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq \frac{3}{2} \end{aligned}$$

*Let  $I \subset \mathbb{R}$  be an interval and consider now a smooth 1-parameter family  $(u_{0,t})_{t \in I}$  of maps  $\mathbb{H}^4 \rightarrow \mathcal{N}$  satisfying the hypothesis of Lemma 3.3. Assume furthermore that*

$$\|\partial_t u_{0,t}\|_{H^3(\mathbb{H}^4; T\mathcal{N})} < \infty$$

*for each  $t \in I$ . Then for each  $t \in I$ , the solution  $u = u_t$  to the harmonic map heat flow with initial data  $u_{0,t}$  given by Lemma 3.2 obeys the following estimates.*

$$\begin{aligned} \|s^{\alpha-1}(-\Delta)^{\alpha+\frac{1}{2}}\partial_t u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} &\lesssim_{\delta, \alpha} \|\partial_t u|_{s=0}\|_{H^3(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq 1 \\ \|s^{\alpha-1}(-\Delta)^{\alpha+\frac{1}{2}}\partial_t u\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} &\lesssim_{\delta, \alpha} \|\partial_t u|_{s=0}\|_{H^3(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha > 1 \\ \|s^{\alpha-1}(-\Delta)^{\alpha-\frac{1}{2}}\partial_t \partial_s u\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} &\lesssim_{\delta, \alpha} \|\partial_t u|_{s=0}\|_{H^3(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq 1 \\ \|s^{\alpha-1}(-\Delta)^{\alpha-\frac{1}{2}}\partial_t \partial_s u\|_{L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} &\lesssim_{\delta, \alpha} \|\partial_t u|_{s=0}\|_{H^3(\mathbb{H}^4; T\mathcal{N})}, \quad \forall \alpha \geq \frac{3}{2} \end{aligned}$$

The proof of Lemma 3.4 is very similar to that of Lemma 3.3 and is omitted. We note that  $\delta_1$  and  $\delta_2$  may need to be adjusted to be slightly smaller.

**3.2. Construction of the Caloric Gauge.** Fix a smooth wave map  $u$  defined on a time interval  $I$  and satisfying the bootstrap assumption (3.3) with initial data satisfying (3.1). Denote by

$$\mathcal{M}_I := I \times \mathbb{H}^4.$$

By (1.7) and finite speed of propagation we can assume that for each  $t \in I$ ,  $u(t, x)$  tends to the same fixed point  $u(t, \infty) \in \mathcal{N}$ , i.e.,

$$\lim_{\mathbf{d}(x, \mathbf{0}) \rightarrow \infty} u(t, x) =: u(t, \infty) = u_\infty \in \mathcal{N} \quad (3.39)$$

In the remainder of the paper, we use the convention that  $d, \nabla, D$  and  $\mathbf{D}$  only refer to spatial derivatives (more precisely, restriction to the tangent space of constant  $t, s$ -surfaces) and use  $d_{t,x}, \nabla_{t,x}, D_{t,x}$  and  $\mathbf{D}_{t,x}$  to denote the corresponding full spacetime derivatives.

The caloric gauge for the map  $u(t, x)$  will be constructed in a sequence of steps.

**Step 1.** In the first step we establish parabolic regularity estimates for the harmonic map heat flow resolution of  $u(t, x)$ . For each fixed  $t \in I$  we use Lemma 3.2 to construct a solution  $u(s, t, \cdot)$  to the harmonic map heat flow with initial data  $u(t, x)$ ,

$$\begin{aligned} \partial_s u^K(t) - \Delta_{\mathbb{H}^4} u^K(t) &= \mathbf{h}^{ab} S_{IJ}^K(u(t)) \partial_a u^I(t) \partial_b u^J(t), \\ u(0, t, x) &= u(t, x). \end{aligned} \quad (3.40)$$

Indeed, we can choose  $\epsilon > 0$  small enough in the bootstrap hypothesis (3.3) so that

$$\sup_{t \in I} \|du(t)\|_{H^1(\mathbb{H}^4; T\mathcal{N})} \leq 2C_0\epsilon < \delta_0 \quad (3.41)$$

where  $\delta_0 = \min\{\delta_1, \delta_2\}$  and  $\delta_1, \delta_2$  are the small constants from Lemma 3.2 and Lemma 3.3. It follows that for each  $t \in I$  we have a unique solution  $u(s, t, x)$  satisfying the bounds (3.10)–(3.14). Moreover, by Lemma 3.3, it follows that the constants in (3.10)–(3.14) can be taken to be uniform in  $t \in I$  and that the map

$$u : \mathbb{R}^+ \times \mathcal{M}_I \rightarrow \mathcal{N} \subset \mathbb{R}^N$$

is smooth. Moreover,  $\partial_t u$  obeys the estimates (3.32)–(3.35).

For simplicity of notation we suppress the dependence of the constants on  $C_0$ . The reader may verify that any appearance of  $C_0$  is accompanied by  $\epsilon$ .

We summarize the  $L^2$  estimates for  $u(s, t, x)$  that we have established above in the following proposition.

**Proposition 3.5.** *Let  $u : \mathcal{M}_I \rightarrow \mathcal{N} \subset \mathbb{R}^N$  be a smooth wave map with initial data satisfying (3.1). Assume the bootstrap assumption (3.3) and let  $u(s, t, x)$  be the harmonic map heat flow resolution of  $u(t, x)$  constructed in Step 1 above. For each fixed  $t \in I$ ,  $du$ ,  $\partial_t u$ , and  $\partial_s u$  satisfy the following estimates:*

$$\begin{aligned} \|(\nabla du, \nabla \partial_t u, \partial_s u, (-\Delta)^{-\frac{1}{2}} \partial_t \partial_s u)(t)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} \\ \lesssim \|(du, \partial_t u)(t)\|_{s=0} \|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \lesssim \epsilon \end{aligned} \quad (3.42)$$

Moreover, for any integer  $k \geq 1$ ,

$$\begin{aligned} \|s^{\frac{k}{2}} (\nabla^{(k+1)} du, \nabla^{(k+1)} \partial_t u, \nabla^{(k)} \partial_s u, \nabla^{(k-1)} \partial_t \partial_s u)(t)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \\ \lesssim \|(du, \partial_t u)(t)\|_{s=0} \|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \lesssim \epsilon \end{aligned} \quad (3.43)$$

Lastly, for any real number  $\alpha > 0$ ,

$$\begin{aligned} \|s^\alpha ((-\Delta)^{\alpha+1} u, (-\Delta)^{\alpha+\frac{1}{2}} \partial_t u)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \\ + \|s^\alpha ((-\Delta)^\alpha \partial_s u, (-\Delta)^{\alpha-\frac{1}{2}} \partial_t \partial_s u)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} \\ \lesssim \|(du, \partial_t u)(t)\|_{s=0} \|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \lesssim \epsilon \end{aligned} \quad (3.44)$$

*Remark 8.* Note that the constants in Proposition 3.5 are independent of  $t$  and  $I$ . Observe also that several of the Sobolev norms in Proposition 3.5 are expressed entirely in terms of the covariant derivative  $\nabla$  on  $\mathbb{H}^4$  in contrast to the estimates in Lemma 3.2 and Lemma 3.3, which involve powers of the Laplacian  $(-\Delta)$ . However, this distinction makes no difference due to the remarks in Section 2.5.

The following pointwise-in- $s$  decay properties of  $u(s, t, x)$  and its derivatives as  $s \rightarrow \infty$  are an easy consequence of Proposition 3.5.

**Lemma 3.6.** *Let  $u(s, t, x)$  be defined as above. Then for any integer  $L \in \mathbb{N}$  we have*

$$\|u(s, t, x) - u_\infty\|_{L_{t,x}^\infty(\mathcal{M}_I)} \lesssim_L \epsilon s^{-L} \quad \text{as } s \rightarrow \infty \quad (3.45)$$

Moreover, we have the following decay estimates for higher derivatives. For each  $j = 1, \dots, N$  and for any integers  $\ell, k, L \geq 0$  we have

$$\|du^j(s, \cdot, \cdot)\|_{L_{t,x}^\infty(\mathcal{M}_I)} \lesssim_L \epsilon s^{-L}, \quad \text{as } s \rightarrow \infty \quad (3.46)$$

$$\|\partial_t u^j(s, \cdot, \cdot)\|_{L_{t,x}^\infty(\mathcal{M}_I)} \lesssim_L \epsilon s^{-L}, \quad \text{as } s \rightarrow \infty \quad (3.47)$$

$$\|\partial_s^k \Delta^\ell u^j(s, \cdot, \cdot)\|_{L_{t,x}^\infty(\mathcal{M}_I)} + \|\partial_s^k \Delta^\ell \partial_t u^j(s, \cdot, \cdot)\|_{L_{t,x}^\infty(\mathcal{M}_I)} \lesssim_L \epsilon s^{-L} \quad (3.48)$$

as  $s \rightarrow \infty$ . Furthermore for any  $\ell, L \in \mathbb{N}$

$$\|D^{(\ell)}(du)(s)\|_{L_{t,x}^\infty} \lesssim_L \epsilon s^{-L} \quad \text{as } s \rightarrow \infty \quad (3.49)$$

where  $D^{(\ell)}$  is the  $\ell$ -th covariant derivative on  $T^*\mathcal{M}_I \otimes u^*T\mathcal{N}$ .

*Proof.* The estimates (3.45), (3.46), and (3.47) are consequences of the Gagliardo-Nirenberg inequality, Sobolev embedding, and the Poincaré inequality on hyperbolic space followed by the parabolic regularity estimates (3.43). For example, for any positive integer  $L \geq 1$  we have

$$\|u(s, \cdot, \cdot) - u_\infty\|_{L_{t,x}^\infty} \lesssim_L s^{-L} \|s^L \Delta^L u(s)\|_{L^2} \lesssim_L \epsilon s^{-L}$$

The estimate (3.48) is proved in the same way as above together with the fact that  $u$  is a smooth solution to

$$\partial_s u = \Delta u + S(u)(du, du)$$

to replace derivatives in  $s$  with spatial derivatives. Finally, we prove (3.49) by repeatedly using the formula

$$D_\beta V = \nabla_\beta V + S(u)(\partial_\beta u, V), \quad V \in \Gamma(u^*T\mathcal{N})$$

together with the estimates (3.46) or (3.47).  $\square$

**Step 2.** The next step is to choose a dynamic frame  $e(s, t, x)$  and associated connection form  $A(s, t, x)$  on the pullback bundle  $u^*T\mathcal{N}$  over  $\mathbb{R}^+ \times \mathcal{M}_I$ . Let  $e$  be a frame on  $u^*T\mathcal{N}$ . Using the isometric embedding  $\mathcal{N} \subset \mathbb{R}^N$ , we may view  $e$  as a collection of  $\mathbb{R}^N$ -valued functions  $(e_j)_{j=1}^n$  which are orthogonal, of length 1, and belong to  $T_u\mathcal{N}$ . As in Section 2.3, we write  $(\partial_s u, \partial_\alpha u) = (e_j \psi_s^j, e_j \psi_\alpha^j)$  and the associated connection 1-form  $A = A_j^i$  on  $\mathbb{R}^+ \times \mathcal{M}_I$  is a traceless anti-symmetric  $n \times n$  matrix-valued 1-form characterized by

$$D_X e_j = A(X)_j^i e_i \quad (3.50)$$

where  $X$  is a vector field on  $T(\mathbb{R}^+ \times \mathcal{M}_I)$ . In coordinates  $((s, x^\alpha), (\frac{\partial}{\partial s}, \frac{\partial}{\partial x^\alpha}))$  on  $T(\mathbb{R}^+ \times \mathcal{M}_I)$  we will have  $A_j^i = A_{j,s}^i ds + A_{j,\alpha}^i dx^\alpha$ , where

$$A_s := A\left(\frac{\partial}{\partial s}\right), \quad A_\alpha := A\left(\frac{\partial}{\partial x^\alpha}\right), \quad \alpha = 0, 1, \dots, 4$$

We denote the curvature in the frame  $e$  by  $F_{\mathbf{ab}} = \partial_{\mathbf{a}} A_{\mathbf{b}} - \partial_{\mathbf{b}} A_{\mathbf{a}} + [A_{\mathbf{a}}, A_{\mathbf{b}}]$ , where  $\mathbf{a}, \mathbf{b} = s, t, x^1, \dots, x^4$ . Recall from Section 2.4 that  $F$  is related to the pullback curvature  $\mathbf{R}$  by

$$F_{\mathbf{ab}} = \mathbf{R}(\psi_{\mathbf{a}}, \psi_{\mathbf{b}})$$

Given such a frame, we can recast 3.5 in terms of  $e$  as

$$\psi_s = \mathbf{h}^{ab} \mathbf{D}_a \psi_b = \text{tr}_{\mathbf{h}} \mathbf{D} \psi$$

Now we define the caloric gauge following Tao [70, 71].

**Definition 1** (Caloric gauge with limiting frame  $e_\infty$ ). Let  $u(s, t, x)$  be the heat flow resolution of  $u(t, x)$  constructed in Step 1 and as in Lemma 3.6. Let  $e_\infty$  be any frame for  $T_{u_\infty} \mathcal{N}$ . A smooth frame  $e(s, t, x)$  is a *caloric gauge* for  $u(s, t, x)$  with *limiting frame*  $e_\infty$  if

- (1) The frame  $e(s, t, x)$  satisfies what is referred to as the *heat-temporal* condition,  $A_s = 0$  for all  $(t, x) \in \mathcal{M}_I$ , or equivalently

$$D_s e = 0. \quad (3.51)$$

- (2) For all  $(t, x) \in \mathcal{M}_I$  we have

$$e(s, t, x) \rightarrow e_\infty \quad \text{as } s \rightarrow \infty. \quad (3.52)$$

**Proposition 3.7** (Existence and Uniqueness of the Caloric Gauge relative to  $e_\infty$ ). *Let  $u(s, t, x)$  be the heat flow resolution of  $u(t, x)$  as in Step 1. Let  $e_\infty$  be any fixed frame for  $T_{u_\infty} \mathcal{N}$ . Then there exists a unique caloric gauge  $e(s, t, x)$  with limiting frame  $e_\infty$ . Moreover, if we denote by  $A_\alpha(s, t, x)$  the associated connection form and  $\psi_s, \Psi = \psi_\alpha dx^\alpha$  the representations of  $\partial_s u$  and  $\partial_\alpha u$  in the frame  $e$  we have*

$$\|\psi_s(s)\|_{L_{t,x}^\infty} \lesssim_L s^{-L} \quad \text{as } s \rightarrow \infty, \quad (3.53)$$

$$\|\Psi(s)\|_{L_{t,x}^\infty} \lesssim_L s^{-L} \quad \text{as } s \rightarrow \infty, \quad (3.54)$$

$$\|A(s)\|_{L_{t,x}^\infty} \lesssim_L s^{-L} \quad \text{as } s \rightarrow \infty, \quad (3.55)$$

for any  $L \in \mathbb{N}$ .

*Proof.* The boundary condition at  $s = \infty$  ensures uniqueness. Indeed, suppose that  $e$  and  $e'$  are both caloric gauges with boundary frame  $e_\infty$ . Then by (3.51) we see that  $|e_j - e'_j|^2$  is constant in  $s$ , vanishes at  $s = \infty$  and is therefore identically 0.

Next we prove existence. We will start with an arbitrary frame at  $s = 0$ , show that it admits a unique extension to  $s \in \mathbb{R}^+$  satisfying the heat-temporal condition, and then perform an  $s$ -independent gauge transform on the extension to ensure that (3.52) holds.

Given any  $(t, x) \in \mathcal{M}_I$  and any choice of smooth frame  $\tilde{e}(0, t, x)$  for  $u(0)^* T\mathcal{N}$ , the heat-temporal gauge condition is imposed by solving the linear ODE (3.51) with initial data  $\tilde{e}(0, t, x)$ . Indeed, as we are viewing  $\mathcal{N} \hookrightarrow \mathbb{R}^N$  as embedded, we solve

$$\begin{aligned} D_s \tilde{e}_j &= \partial_s \tilde{e}_j + S(u)(\tilde{e}_j, \partial_s u) = 0 \\ \tilde{e}_j|_{s=0} &= \tilde{e}_j(0, \cdot, \cdot) \end{aligned} \quad (3.56)$$

for each  $j = 1, \dots, n$ . We find using Picard iteration a global, smooth, unique solution  $\tilde{e}(s, t, x)$  to (3.56). Moreover  $\tilde{e}(s, t, x)$  is a frame for  $u(s)^*T\mathcal{N}$  for each  $s \in \mathbb{R}^+$  since we have

$$\partial_s \langle \tilde{e}_j, \tilde{e}_k \rangle_{u^*T\mathcal{N}} = 2 \langle D_s \tilde{e}_j, \tilde{e}_k \rangle_{u^*T\mathcal{N}} = 0 \quad (3.57)$$

and thus  $\langle \tilde{e}_j, \tilde{e}_k \rangle_{u^*T\mathcal{N}} = \delta_{jk}$  for all  $j, k \in \{1, \dots, n\}$ . Next, using the equation (3.56), for any  $L \in \mathbb{N}$ ,

$$\|\partial_s \tilde{e}_j(s)\|_{L_{t,x}^\infty} = \|S(u)(\tilde{e}_j, \partial_s u)\|_{L_{t,x}^\infty} \lesssim \|\partial_s u(s)\|_{L_{t,x}^\infty} \lesssim_L s^{-L}.$$

This implies that  $\tilde{e}(s, t, x)$  converges to a limiting frame  $\tilde{e}_\infty(t, x)$  as  $s \rightarrow \infty$ . Now, let  $\tilde{A}_\alpha$ , denote the connection form associated to  $\tilde{e}$  and let  $\tilde{\psi}_s$  and  $\tilde{\Psi} = \tilde{\psi}_\alpha dx^\alpha$  denote the representations of  $\partial_s s$  and  $\partial_\alpha u$  in the frame  $\tilde{e}$ . Note that

$$|\tilde{\psi}_s| = |\tilde{e}\tilde{\psi}_s| = |\partial_s u| \lesssim_L s^{-L}, \quad |\tilde{\Psi}| = |\tilde{e}\tilde{\Psi}| = |(du, \partial_t u)| \lesssim_L s^{-L} \quad (3.58)$$

Next we claim that

$$|\tilde{F}_{s\alpha}| \lesssim |\partial_s u| |\partial_\alpha u| \lesssim_L s^{-L} \quad (3.59)$$

where we recall that  $|F_{s\alpha}(s, t, x)|$  is the Hilbert-Schmidt norm for  $n \times n$  matrices. Indeed (3.59) is a direct consequence of the bounded geometry of  $\mathcal{N}$  and the identity

$$\tilde{e}\tilde{F}_{s\alpha} = R(u)(\partial_s u, \partial_\alpha u)\tilde{e} \quad (3.60)$$

Now note that by (3.56) we have

$$0 = D_\alpha D_s \tilde{e} = D_s D_\alpha \tilde{e} + \tilde{e}\tilde{F}_{\alpha s} \quad (3.61)$$

Since  $\tilde{A}_s = 0$  we have  $D_s = \partial_s$  and thus

$$|\partial_s D_\alpha \tilde{e}| = |\tilde{F}_{s\alpha}| \lesssim |\partial_s u| |\partial_\alpha u| \lesssim_L s^{-L}. \quad (3.62)$$

It follows that

$$D_\alpha \tilde{e} \rightarrow D_\alpha \tilde{e}_\infty \quad \text{as } s \rightarrow \infty, \quad \alpha = 0, 1, \dots, 4 \quad (3.63)$$

Again since  $\tilde{A}_s = 0$  we then have  $\partial_s \tilde{A}_\alpha = \tilde{F}_{s\alpha}$  and thus

$$|\partial_s \tilde{A}_\alpha| = |\tilde{F}_{s\alpha}| \lesssim |\partial_s u| |\partial_\alpha u| \lesssim_L s^{-L} \quad \text{as } s \rightarrow \infty, \quad \alpha = 0, 1, \dots, 4 \quad (3.64)$$

This implies that each  $\tilde{A}_\alpha(s, t, x)$  converges to a limit  $\tilde{A}_\alpha(t, x)$ ; more succinctly

$$\tilde{A}(s, t, x) \rightarrow \tilde{A}_\infty(t, x) \in \mathfrak{so}(n) \quad \text{as } s \rightarrow \infty. \quad (3.65)$$

Combining the above with (3.63) we can pass to limits in both sides of

$$\tilde{A}_{j,\alpha}^k = \langle D_\alpha \tilde{e}_j, \tilde{e}_k \rangle \quad (3.66)$$

to deduce that  $\tilde{A}_\infty$  is the connection form for  $\tilde{e}_\infty$ .

Now find a gauge transformation  $B(t, x) \in SO(n)$  so that

$$\tilde{e}_\infty(t, x)B(t, x) = e_\infty \quad (3.67)$$

The caloric gauge  $e$  is then defined by applying  $B(t, x)$  as above at each time  $s$ , i.e.,

$$e(s, t, x) := \tilde{e}(s, t, x)B(t, x), \quad e_j(s) := \tilde{e}_\ell(s)B_j^\ell. \quad (3.68)$$

This ensures that for any fixed  $(t, x) \in \mathcal{M}_I$ , we have the convergence,

$$e(s, t, x) \rightarrow e_\infty \quad \text{as } s \rightarrow \infty \quad (3.69)$$

Let  $A(s, t, x)$  denote the connection form associated to  $e(s, t, x)$ . Arguing as above and using (3.69) we must have

$$A(s, t, x) \rightarrow 0 \quad \text{as } s \rightarrow \infty \quad (3.70)$$

since the associated connection 1-form to the constant frame  $e_\infty$  on  $\mathcal{M}_I$  is  $A_\infty = 0$ . To deduce (3.55) regarding the rate of this convergence, we again argue as above to see that  $\partial_s A_\alpha = F_{s\alpha}$  and that

$$|\tilde{F}_{s\alpha}| \lesssim |\partial_s u| |\partial_\alpha u| \lesssim_L s^{-L} \quad (3.71)$$

Hence

$$|A_\alpha(s)| \leq \int_s^\infty |F_{s\alpha}(s')| ds' \lesssim_{L+1} s^{-L} \quad \text{as } s \rightarrow \infty, \quad \alpha = 0, 1, \dots, 4 \quad (3.72)$$

Finally we note that (3.53) and (3.54) follow from the exact same argument as in (3.58).  $\square$

**3.3. Summary of the dynamic variables in the caloric gauge.** Putting the contents of the previous subsection together we record a list of the dynamic variables that have been introduced starting from a wave map  $(u(t, x), \partial_t u(t, x))$  on a time interval  $I$ , finding its harmonic heat flow resolution  $u(s, t, x)$ , and passing to the caloric gauge  $e$  with boundary frame  $e_\infty$ . Below  $\alpha \in \{0, 1, \dots, 4\}$ ,  $a \in \{1, \dots, 4\}$ ,  $j \in \{1, \dots, n\}$ .

$$\partial_s u := e\psi_s = \psi_s^j e_j, \quad \partial_\alpha u = e\psi_\alpha, \quad (3.73)$$

$$\psi_s^j(s, t, x) = \mathbf{D}^a \psi_a^j(s, t, x) := \langle e_k \mathbf{h}^{ab} (\mathbf{D}\psi)_{ab}^k, e_j \rangle_{u^* \mathbf{g}}, \quad (3.74)$$

$$\mathbf{D}^\alpha \psi_\alpha^j(0, t, x) := \langle e_k \boldsymbol{\eta}^{\alpha\beta} (\mathbf{D}\psi)_{\alpha\beta}^k(0, t, x), e_j \rangle_{u^* \mathbf{g}} = 0, \quad (3.75)$$

$$D_\alpha e =: eA_\alpha, \quad (3.76)$$

$$D_s e =: eA_s = 0, \quad (3.77)$$

$$\mathbf{D}_\alpha \psi_\beta = \mathbf{D}_\beta \psi_\alpha, \quad (3.78)$$

$$\partial_s \psi_\alpha = \mathbf{D}_s \psi_\alpha = \mathbf{D}_\alpha \psi_s, \quad (3.79)$$

$$\partial_s A_\alpha = F_{s\alpha}, \quad F_{\alpha\beta} = \mathbf{R}^{(0)}(\psi_\alpha, \psi_\beta), \quad eF_{\alpha\beta} = R(u)(\partial_\alpha u, \partial_\beta u)e. \quad (3.80)$$

In the remainder of the paper, it will be convenient to have separate notation for the spatial and temporal components of tensors. For the space-time 1-forms  $\psi_\alpha$ ,  $A_\alpha$ , and  $F_{s\alpha}$  we introduce the following notation:

$$\begin{aligned} \Psi &= \psi_\alpha dx^\alpha = \psi_t dt + \underline{\Psi}, \quad \underline{\Psi} = \psi_a dx^a \\ A &= A_\alpha dx^\alpha = A_t dt + \underline{A}, \quad \underline{A} = A_a dx^a \\ F_s &= F_{s\alpha} dx^\alpha = F_{st} dt + \underline{F}_s, \quad \underline{F}_s = F_{sa} dx^a \end{aligned} \quad (3.81)$$

We remind the reader that the pointwise norm of space-time one forms is defined with the auxiliary metric  $\tilde{\boldsymbol{\eta}}$ ; see (2.37). Moreover, for differential operators, we use the convention that  $d$ ,  $\nabla$ , and  $\mathbf{D}$  refer only to spatial components, and  $d_{t,x}$ ,  $\nabla_{t,x}$ , and  $\mathbf{D}_{t,x}$  refer to the whole spatial and temporal components.

We also make note of the boundary conditions

$$\begin{aligned} e(s, t, x) &\rightarrow e_\infty \quad \text{as } s \rightarrow +\infty \\ \psi_s(s, t, x), \psi_\alpha(s, t, x), A_\alpha(s, t, x) &\rightarrow 0 \quad \text{as } s \rightarrow +\infty \end{aligned} \quad (3.82)$$

Crucially, the above allow us to express  $A_\alpha$  and  $\psi_\alpha$  in terms of  $\psi_s$  and  $F_{s\alpha}$  by the fundamental theorem of calculus,

$$A_\alpha(s_0, t, x) = - \int_{s_0}^{\infty} \partial_s A_\alpha(s, t, x) ds = - \int_{s_0}^{\infty} F_{s\alpha}(s) ds \quad (3.83)$$

and

$$\begin{aligned} \psi_\alpha(s_0, t, x) &= - \int_{s_0}^{\infty} \partial_s \psi_\alpha ds = - \int_{s_0}^{\infty} \mathbf{D}_\alpha \psi_s ds \\ &= - \int_{s_0}^{\infty} \partial_\alpha \psi_s ds - \int_{s_0}^{\infty} A_\alpha \psi_s ds \\ &= - \int_{s_0}^{\infty} \partial_\alpha \psi_s ds + \int_{s_0}^{\infty} \int_s^{\infty} F_{s\alpha}(s') \psi_s(s) ds' ds. \end{aligned} \quad (3.84)$$

To estimate the above we will often use the pointwise bound

$$|F_{s\alpha}(s)| \lesssim |\partial_s u(s)| |(du, \partial_t u)(s)| \quad (3.85)$$

which follows from the identity

$$e F_{s\alpha} = R(u)(\partial_s u, \partial_\alpha u)e$$

and the bounded geometry of  $\mathcal{N}$ .

**3.4. Equivalence of norms in the caloric gauge.** We now transfer the  $L^2$ -based parabolic regularity estimates for  $\partial_\alpha u$  and  $\partial_s u$  from Section 3.2 to  $L^2$  based estimates for  $\psi_s$  and  $\psi_\alpha$ .

To begin, let  $\phi$  be any smooth section of  $u^* T\mathcal{N}$ , and let  $e$  be any frame for  $u^* T\mathcal{N}$ . Set  $\phi =: e\varphi$ . Then,

$$\|\varphi\|_{L^2(\mathbb{H}^4; \mathbb{R}^n)} = \|\phi\|_{L^2(\mathbb{H}^4; \mathbb{R}^N)}.$$

It follows then, for example, that

$$\|\psi_s(s, t)\|_{L^2(\mathbb{H}^4; \mathbb{R}^n)} = \|\partial_s u(s, t)\|_{L^2(\mathbb{H}^4; \mathbb{R}^N)}.$$

To prove similar equivalence of norms statements in the caloric gauge  $e$ , for higher derivatives of  $\partial_s u$ , and  $\partial_\alpha u$  we will use the formula

$$D_\beta \phi = \nabla_\beta \phi + S(u)(\partial_\beta u, \phi) = e(\nabla_\beta \varphi + A_\beta \varphi). \quad (3.86)$$

We start with the following proposition.

**Proposition 3.8** (Equivalence of norms). *Let  $u(s, t, x)$  be the heat flow resolution of  $u(t, x)$  as in Proposition 3.5. Let  $\psi_\alpha$  and  $\psi_s$  be the representations of  $\partial_\alpha u$  and  $\partial_s u$  in the caloric gauge  $e$  with limiting frame  $e_\infty$  as in Proposition 3.7. Then for each  $t \in I$ ,*

$$\begin{aligned} \|\nabla \Psi(t)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2(\mathbb{H}^4; \mathbb{R}^N))} &\simeq \|(\nabla du, \nabla \partial_t u)(t)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2(\mathbb{H}^4; \mathbb{R}^N))} \\ &\simeq \|Dd_{t,x} u(t)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2(\mathbb{H}^4; \mathbb{R}^N))} \end{aligned} \quad (3.87)$$

and

$$\begin{aligned} \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s(t)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2(\mathbb{H}^4; \mathbb{R}^n))} &\simeq \|s^{\frac{1}{2}} \nabla_{t,x} \partial_s u(t)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2(\mathbb{H}^4; \mathbb{R}^N))} \\ &\simeq \|s^{\frac{1}{2}} D_{t,x} \partial_s u(t)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2(\mathbb{H}^4; \mathbb{R}^N))} \end{aligned} \quad (3.88)$$

where the implicit constants are independent of  $t$  and  $I$ .

From (3.86) it is clear that to compare, say  $\nabla \partial_\alpha u$  with  $\nabla \psi_\alpha$ , we will need control over the connection form  $A$ . We require the following lemma.

**Lemma 3.9.** *Let  $u(s, t, x)$  be the heat flow resolution of  $u(t, x)$  as in Proposition 3.5. Let  $e$  be the caloric gauge with limiting frame  $e_\infty$  as in Proposition 3.7, and let  $A$  denote the corresponding connection form. Then for each  $t \in I$  we have*

$$\|A(t)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^4(\mathbb{H}^4; \mathfrak{so}(n)))} \lesssim \|(du, \partial_t u)(t)\rfloor_{s=0}\|_{H^1(\mathbb{H}^4; T\mathcal{N})}^2 \lesssim \epsilon^2 \quad (3.89)$$

We also have the following estimates on the derivatives of  $A$  and  $\underline{A}$ .

$$\|(\nabla A, \nabla_{t,x} \underline{A})(t)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2(\mathbb{H}^4; \mathfrak{so}(n)))} \lesssim \|(du, \partial_t u)(t)\rfloor_{s=0}\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})}^2 \lesssim \epsilon^2 \quad (3.90)$$

Moreover, for all  $k \geq 1$ , we have,

$$\begin{aligned} \|s^{\frac{k}{2}} (\nabla^{(k+1)} A, \nabla^{(k)} \nabla_{t,x} \underline{A})(t)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2(\mathbb{H}^4; \mathfrak{so}(n)))} \\ \lesssim_k \|(du, \partial_t u)(t)\rfloor_{s=0}\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})}^2 \lesssim \epsilon^2 \end{aligned} \quad (3.91)$$

All constants are independent of  $t$  and  $I$ .

Given Proposition 3.8 and Lemma 3.9 we can also deduce the following bounds on higher derivatives of  $\Psi$  and  $\psi_s$ .

**Corollary 3.10.** *Let  $u(s, t, x)$  be the heat flow resolution of  $u(t, x)$  as in Proposition 3.5. Let  $\psi_\alpha$  and  $\psi_s$  be the representations of  $\partial_\alpha u$  and  $\partial_s u$  in the caloric gauge  $e$  with limiting frame  $e_\infty$  as in Proposition 3.7. Then for each  $t \in I$  and all  $k \geq 2$*

$$\begin{aligned} \|s^{\frac{k}{2}} D^{(k-1)} D_{t,x} \partial_s u(t)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \lesssim_k \|(du, \partial_t u)(t)\rfloor_{s=0}\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \\ \|s^{\frac{k}{2}} \nabla^{(k-1)} \nabla_{t,x} \psi_s(t)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \lesssim_k \|(du, \partial_t u)(t)\rfloor_{s=0}\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \end{aligned} \quad (3.92)$$

and for each  $t \in I$  and  $k \geq 1$ ,

$$\begin{aligned} \|s^{\frac{k}{2}} D^{(k+1)} d_{t,x} u(t)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \lesssim_k \|(du, \partial_t u)(t)\rfloor_{s=0}\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \\ \|s^{\frac{k}{2}} \nabla^{(k+1)} \Psi(t)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; L^2)} \lesssim_k \|(du, \partial_t u)(t)\rfloor_{s=0}\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \end{aligned} \quad (3.93)$$

We first prove Lemma 3.9.

*Proof of Lemma 3.9.* Fix  $t \in I$ ; in what follows, we suppress writing  $t$  with the understanding that every computation is performed at the fixed time  $t$ . By (3.83) we have  $A_\alpha(s_0) = -\int_{s_0}^\infty F_{s\alpha}(s) ds$  for any  $s_0 > 0$ . It follows from (3.85) that

$$\begin{aligned} \|A(s_0)\|_{L^4} &\lesssim \int_{s_0}^\infty \|F_s(s)\|_{L^4} ds \lesssim \int_{s_0}^\infty \|\partial_s u\|_{L^4} \|d_{t,x} u\|_{L^\infty} ds \\ &\lesssim \left( \int_{s_0}^\infty s \|\partial_s u\|_{L^4}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \left( \int_{s_0}^\infty s \|d_{t,x} u\|_{L^\infty}^2 \frac{ds}{s} \right)^{\frac{1}{2}}. \end{aligned}$$



By Sobolev embedding the first integral on the right above is bounded using (3.43) with  $k = 1$ , by

$$\begin{aligned} \left( \int_{s_0}^{\infty} s \|\partial_s u\|_{L^4}^2 \frac{ds}{s} \right)^{\frac{1}{2}} &\lesssim \left( \int_{s_0}^{\infty} \|s^{\frac{1}{2}} \nabla \partial_s u\|_{L^2}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\lesssim \|(du, \partial_t u)(t)\}|_{s=0} \|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \lesssim \epsilon \end{aligned}$$

To bound the second integral we use Gagliardo-Nirenberg, Sobolev and  $L^2$  interpolation,

$$\begin{aligned} \left( \int_{s_0}^{\infty} s \|du\|_{L^\infty}^2 \frac{ds}{s} \right)^{\frac{1}{2}} &\lesssim \left( \int_{s_0}^{\infty} s \|du\|_{L^8} \|\nabla du\|_{L^8} \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\lesssim \left( \int_{s_0}^{\infty} \|s^{\frac{1}{4}} (-\Delta)^{\frac{5}{4}} u\|_{L^2} \|s^{\frac{3}{4}} (-\Delta)^{\frac{7}{4}} u\|_{L^2} \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\lesssim \|(du, \partial_t u)(t)\}|_{s=0} \|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \lesssim \epsilon \end{aligned}$$

where the last line follows from (3.43) and (3.44). The contribution of  $\partial_t u$  to the second integral can be estimated in a similar way.

To prove the second bound we begin with the formula

$$\begin{aligned} \nabla_\beta A_\alpha(s_0) &= - \int_{s_0}^{\infty} \nabla_\beta F_{s\alpha}(s) ds \\ &= - \int_{s_0}^{\infty} \mathbf{D}_\beta F_{s\alpha}(s) ds + \int_{s_0}^{\infty} [A_\beta, F_{s\alpha}](s) ds \end{aligned}$$

where at most one of  $\alpha$  and  $\beta$  is 0. By (2.24) and (2.25) in Section 2.4, as well as  $\partial_s u = e\psi_s$  and  $\partial_\alpha u = e\psi_\alpha$ , the term  $\mathbf{D}_\beta F_{s\alpha}$  is pointwisely bounded by

$$|\mathbf{D}_\beta F_{s\alpha}| \lesssim |D_\beta \partial_s u| |\partial_\alpha u| + |\partial_s u| |D_\beta \partial_\alpha u| + |\partial_\beta u| |\partial_s u| |\partial_\alpha u|$$

Using  $D_\beta \partial_\alpha u = D_\alpha \partial_\beta u$  if necessary, it follows that

$$\begin{aligned} |\mathbf{D}_{t,x} F_s| + |[A, F_s]| &\lesssim |\nabla_{t,x} \partial_s u| |d_{t,x} u| + |\partial_s u| |\nabla d_{t,x} u| \\ &\quad + |\partial_s u| |d_{t,x} u|^2 + |A| |\partial_s u| |d_{t,x} u| \end{aligned}$$

Therefore

$$\begin{aligned} &\|\nabla_{t,x} A\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+, L^2)} \\ &\lesssim \int_0^\infty s \|\nabla_{t,x} \partial_s u\|_{L^{\frac{8}{3}}} \|d_{t,x} u\|_{L^8} \frac{ds}{s} + \int_0^\infty s \|\partial_s u\|_{L^4} \|\nabla d_{t,x} u\|_{L^4}^2 \frac{ds}{s} \\ &\quad + \int_0^\infty s \|\partial_s u\|_{L^4} \|d_{t,x} u\|_{L^8}^2 \frac{ds}{s} + \int_0^\infty s \|A\|_{L^4} \|\partial_s u\|_{L^8} \|d_{t,x} u\|_{L^8} \frac{ds}{s} \\ &\lesssim \|(du, \partial_t u)(t)\}|_{s=0} \|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \lesssim \epsilon, \end{aligned}$$

where the last line is a consequence of Sobolev embedding, Gagliardo-Nirenberg, the estimates in Proposition 3.5 and (3.89). This completes the proof of (3.90). One can argue similarly to establish (3.91).  $\square$

Next, we establish Proposition 3.8.

*Proof of Proposition 3.8.* As before, we fix and suppress  $t \in I$ . We first prove (3.87). To prove the first  $\simeq$  in (3.87) we deduce from (3.86) with  $\phi = \partial_\alpha u$  and  $\varphi = \psi_\alpha$  that

$$\begin{aligned} & \left| \|(\nabla du, \nabla \partial_t u)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} - \|\nabla \Psi\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} \right| \\ & \leq \|S(u)(du, e\Psi)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} + \|S(u)(\partial_t u, e\Psi)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} + \|Ae\Psi\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)} \\ & \lesssim \left( \|(du, \partial_t u)\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^4)} + \|A\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^4)} \right) \|\Psi\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^4)} \\ & \lesssim \epsilon \|\nabla \Psi\|_{L_{\frac{ds}{s}}^\infty(\mathbb{R}^+; L^2)}. \end{aligned}$$

A similar argument gives the second  $\simeq$  in (3.87). The proof of (3.88) is identical after including the appropriate factor of  $s^{\frac{1}{2}}$ .  $\square$

Finally, we give a brief sketch of the proof of Corollary 3.10.

*Proof of Corollary 3.10.* The proof is essentially the same as the proof of Proposition 3.8 and is based on equation (3.86). For example, to prove (3.92) with  $k = 2$  we apply (3.86) with  $\phi = D_\alpha \partial_s u$ . We have,

$$\begin{aligned} D_b(D_\alpha \partial_s u) &= \nabla_b \nabla_\alpha \partial_s u + \nabla_b(S(u)(\partial_s u, \partial_\alpha u)) + S(u)(D_\alpha \partial_s u, \partial_b u) \\ &= e \nabla_b \nabla_\alpha \psi_s + e \nabla_b(A_\alpha \psi_s) + e A_b \nabla_\alpha \psi_s + e A_b A_\alpha \psi_s. \end{aligned}$$

One can then argue as in the proof of Proposition 3.8 using the bounds from Proposition 3.5, Lemma 3.9, and (3.88) to conclude.  $\square$

**3.5. The wave map system in the caloric gauge: Dynamic equations for  $\psi_s$  and  $\psi_\alpha$ .** Given the estimates in the previous subsection, where we transferred the  $L^2$ -based parabolic regularity estimates for the harmonic map heat flow resolution  $u(s, t, x)$  to the representations of  $\partial_\alpha u$  and  $\partial_s u$  in the caloric gauge, namely  $\psi_\alpha$  and  $\psi_s$ , we can now work with the *dynamic* equations for  $\psi_\alpha$  and  $\psi_s$  to close the a priori estimates (3.2). We place particular importance on the *scalar* nonlinear wave equation (3.96) satisfied by  $\psi_s$  since one can *close* a priori estimates for solutions to (3.96) using standard analysis based on Strichartz estimates.

Following [70], we refer to  $\psi_s$  as the *heat tension field*,

$$\psi_s(s, t, x) := \mathbf{D}^a \psi_a \quad (3.94)$$

We also define the *wave tension field*  $w$  by

$$w(s, t, x) := \mathbf{D}^\alpha \psi_\alpha \quad (3.95)$$

We note that since  $u(0, t, x)$  is a wave map we have  $w(0, t, x) = 0$ .

We derive a covariant wave equation for the heat tension field  $\psi_s$  and a covariant heat equation for the wave tension field  $w$ . A key observation is that both equations are *scalar*, in the sense that the linear parts coincide with the scalar wave and heat equations on  $\mathbb{H}^4$ , respectively.

**Lemma 3.11** (Dynamic equations in the caloric gauge: wave equation for  $\psi_s$ ). *Let  $\psi_s$  and  $w$  be defined as in (3.95) and (3.94). Then the heat tension field  $\psi_s$  satisfies*

$$\mathbf{D}^\alpha \mathbf{D}_\alpha \psi_s = \partial_s w - F_s{}^\alpha \psi_\alpha \quad (3.96)$$

where the wave tension field  $w$  satisfies

$$\begin{aligned} (\partial_s - \mathbf{D}^b \mathbf{D}_b)w &= F_s^\alpha \psi_\alpha + 3F^{\alpha b} \mathbf{D}_\alpha \psi_b + \mathbf{D}^\alpha F_\alpha^b \psi_b + \mathbf{D}^b F_b^\alpha \psi_\alpha \\ w(0, t, x) &= 0 \end{aligned} \quad (3.97)$$

*Remark 9.* In order to estimate  $w(s)$ , we will solve the inhomogeneous heat equation (3.97) from 0 to  $s$ , exploiting the fact that the initial data for  $w$  is zero. Observe that all other terms in (3.96) only involve variables at  $s$  or larger. In the Littlewood-Paley analogy discussed in Section 1.1,  $\partial_s w$  can be thought of as the contribution of Littlewood-Paley projections at higher frequencies.

*Proof.* First we prove (3.96). Using the identity (3.79), the fact that  $\mathbf{D}_s = \partial_s$  and that the domain curvature  $\mathcal{R}_{\mu_2 \mu_3 \mu_4}^{\mu_1}$  vanishes if any of the indices  $\mu_j = s$  we have

$$\mathbf{D}^\alpha \mathbf{D}_\alpha \psi_s = \mathbf{D}^\alpha \mathbf{D}_s \psi_\alpha = F_s^\alpha \psi_\alpha + \partial_s w$$

Next, we derive the covariant heat equation (3.97) for the heat tension field  $w$  making use of the identities (3.78) and (3.79). We have

$$\begin{aligned} \partial_s w &= \partial_s \mathbf{D}^\alpha \psi_\alpha = \mathbf{D}^\alpha \mathbf{D}_s \psi_\alpha + [\mathbf{D}_s, \mathbf{D}^\alpha] \psi_\alpha \\ &= \mathbf{D}^\alpha \mathbf{D}_\alpha \mathbf{D}^b \psi_b + F_s^\alpha \psi_\alpha \\ &= \mathbf{D}^\alpha \mathbf{D}^b \mathbf{D}_b \psi_\alpha + \mathbf{D}^\alpha [\mathbf{D}_\alpha, \mathbf{D}^b] \psi_b + F_s^\alpha \psi_\alpha \\ &= \mathbf{D}^b \mathbf{D}_b w + F_s^\alpha \psi_\alpha + [\mathbf{D}^\alpha, \mathbf{D}^b] \mathbf{D}_b \psi_\alpha + \mathbf{D}^b [\mathbf{D}^\alpha, \mathbf{D}_b] \psi_\alpha + \mathbf{D}^\alpha [\mathbf{D}_\alpha, \mathbf{D}^b] \psi_b \end{aligned}$$

By the commutation formula (2.16), the last three terms take the form

$$\begin{aligned} &[\mathbf{D}^\alpha, \mathbf{D}^b] \mathbf{D}_b \psi_\alpha + \mathbf{D}^b [\mathbf{D}^\alpha, \mathbf{D}_b] \psi_\alpha + \mathbf{D}^\alpha [\mathbf{D}_\alpha, \mathbf{D}^b] \psi_b \\ &= F^{\alpha b} \mathbf{D}_b \psi_\alpha + \mathcal{R}^{ab}{}^c{}_b \mathbf{D}_c \psi_a + \mathcal{R}^{ab}{}^a{}_b \mathbf{D}_b \psi_c + \mathbf{D}^b (F_b^\alpha \psi_\alpha + \mathcal{R}_{ba}^a{}^c \psi_c) \\ &\quad + \mathbf{D}^\alpha (F_\alpha^b \psi_b) + \mathbf{D}^a (\mathcal{R}_a{}^b{}^c{}_b \psi_c) \end{aligned}$$

where we used the fact that the domain curvature vanishes if any of the indices equal to 0 to replace  $\alpha$  by  $a$ . By the anti-symmetry properties of the curvature tensor  $\mathcal{R}$ , note that the terms involving  $\mathcal{R}$  all cancel. Plugging this into the equation for  $\partial_s w$ , we obtain (3.97).  $\square$

**Lemma 3.12** (Heat equation for  $\psi_s$ ). *The heat tension field  $\psi_s$  satisfies the following linearized covariant heat equation in  $s$ :*

$$\partial_s \psi_s - \mathbf{D}^b \mathbf{D}_b \psi_s = F_s^b \psi_b = \mathbf{R}^{(0)}(\psi_s, \psi^b) \psi_b \quad (3.98)$$

*Proof.* Since any component of the domain curvature tensor  $\mathcal{R}$  on  $\mathbb{R}^+ \times \mathcal{M}_I$  with an  $s$  index is 0, we have

$$\begin{aligned} \partial_s \psi_s &= \partial_s \mathbf{D}^b \psi_b = \mathbf{D}_s \mathbf{D}^b \psi_b = \mathbf{D}^b \mathbf{D}_s \psi_b + F_s^b \psi_b \\ &= \mathbf{D}^b \mathbf{D}_b \psi_s + F_s^b \psi_b \end{aligned}$$

which establishes (3.98).  $\square$

**Lemma 3.13** (Covariant heat equations for  $\psi_a$ ). *Let  $\psi_t, \psi_a$  denote the representations of  $\partial_t u$  and  $\partial_a u$  in the caloric gauge  $e$ , where  $a \in \{1, \dots, 4\}$  is a spatial index. Then  $\psi_t$  satisfies the following linearized covariant heat equation*

$$(\partial_s - \mathbf{D}^b \mathbf{D}_b) \psi_t = F_t^b \psi_b = \mathbf{R}^{(0)}(\psi_t, \psi^b) \psi_b \quad (3.99)$$

and  $\psi_a$  satisfies the tensorial covariant heat equation

$$(\partial_s - \mathbf{D}^b \mathbf{D}_b - 3)\psi_a = F_a{}^b \psi_b = \mathbf{R}^{(0)}(\psi_a, \psi^b) \psi_b \quad (3.100)$$

We remark that the linear part of (3.99) coincides with the scalar linear heat equation where  $\psi_t$  is viewed as a scalar function on  $\mathbb{H}^4$ , whereas the linear part of (3.100) is tensorial.

*Proof.* With  $\alpha \in \{0, 1, \dots, 4\}$  we have

$$\begin{aligned} \partial_s \psi_\alpha &= \mathbf{D}_s \psi_\alpha = \mathbf{D}_\alpha \psi_s = \mathbf{D}_\alpha \mathbf{D}^b \psi_b \\ &= \mathbf{D}^b \mathbf{D}_\alpha \psi_b + [\mathbf{D}_\alpha, \mathbf{D}_b] \mathbf{h}^{bc} \psi_c \\ &= \mathbf{D}^b \mathbf{D}_b \psi_\alpha + [\mathbf{D}_\alpha, \mathbf{D}_b] \mathbf{h}^{bc} \psi_c \end{aligned}$$

As in Lemma 3.12, the components of the domain curvature tensor  $\mathcal{R}$  on  $\mathbb{R}^+ \times \mathcal{M}_I$  all vanish when  $\alpha = 0$ . Hence, in that case we have

$$[\mathbf{D}_t, \mathbf{D}_b] \mathbf{h}^{bc} \psi_c = F_t{}^b \psi_b$$

When  $\alpha = a \in \{1, \dots, 4\}$  we have

$$\begin{aligned} [\mathbf{D}_a, \mathbf{D}_b] \mathbf{h}^{bc} \psi_c &= F_a{}^b \psi_b - \mathcal{R}_{eba} \mathbf{h}^{ec} \psi_c = F_a{}^b \psi_b - \mathcal{R}_{ea} \mathbf{h}^{ec} \psi_c \\ &= F_a{}^b \psi_b + 3\mathbf{h}_{ae} \mathbf{h}^{ec} \psi_c = F_a{}^b \psi_b + 3\psi_a \end{aligned}$$

which completes the proof.  $\square$

We conclude with identities regarding the spacetime and space divergences of  $F$ . These identities are useful for handling the last two terms on the right-hand side of (3.97), as well as dealing with the term  $\nabla^\alpha A_\alpha$  that arise from expanding the covariant d'Alembertian  $\mathbf{D}^\alpha \mathbf{D}_\alpha$ .

**Lemma 3.14** (Spacetime and space divergences of  $F$ ). *Let  $F_{\mathbf{a}\beta}$  denote a component of the curvature 2-form of a wave map in the caloric gauge  $e$ , where  $\mathbf{a}$  denotes any one of the coordinates  $(s, x^\alpha)$  and  $\beta = 0, 1, \dots, 4$ . Then  $F_{\mathbf{a}\beta}$  satisfies the following covariant divergence identities:*

$$\mathbf{D}^\beta F_{\mathbf{a}\beta} = \mathbf{R}^{(0)}(\mathbf{D}^\beta \psi_{\mathbf{a}}, \psi_\beta) + \mathbf{R}^{(0)}(\psi_{\mathbf{a}}, w) + \mathbf{R}^{(1)}(\psi^\beta; \psi_{\mathbf{a}}, \psi_\beta) \quad (3.101)$$

$$\mathbf{D}^b F_{ab} = \mathbf{R}^{(0)}(\mathbf{D}^b \psi_{\mathbf{a}}, \psi_b) + \mathbf{R}^{(0)}(\psi_{\mathbf{a}}, \psi_s) + \mathbf{R}^{(1)}(\psi^b; \psi_{\mathbf{a}}, \psi_b) \quad (3.102)$$

*Proof.* These identities follow from (2.24) and the definition of  $w$  and  $\psi_s$ .  $\square$

#### 4. RE-STATEMENT OF THE MAIN THEOREM AND OUTLINE OF THE STRATEGY

Now that we have constructed the caloric gauge we can restate Theorem 1.1, this time including a priori estimates on dispersive norms of  $\psi_s$ .

Let  $I \subset \mathbb{R}$  be a time interval and let  $v = v(s, t, x)$  be a smooth function  $v : \mathbb{R}^+ \times I \times \mathbb{H}^4 \rightarrow \mathbb{R}^n$ . For each  $s > 0$  define

$$\begin{aligned} \|v(s)\|_{\mathcal{S}_s(I)} &:= \|s^{\frac{1}{2}} v(s)\|_{L_t^2(I; L_x^8(\mathbb{H}^4; \mathbb{R}^n))} + \|s^{\frac{1}{2}} (-\Delta)^{-\frac{1}{2}} \partial_t v\|_{L_t^2(I; L_x^8(\mathbb{H}^4; \mathbb{R}^n))} \\ &\quad + \|s^{\frac{1}{2}} \nabla_{t,x} v(s)\|_{L_t^\infty(I; L_x^2(\mathbb{H}^4; \mathbb{R}^n))}. \end{aligned} \quad (4.1)$$

We then define the norm  $\mathcal{S}(I)$  by

$$\|v\|_{\mathcal{S}(I)} := \|v(s)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 \mathcal{S}_s(I)} \quad (4.2)$$

where we have used the slight abuse of notation  $L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 X := L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(\mathbb{R}^+; X)$  for any space(-time) norm  $X$ , which will be in effect for the rest of this paper.

**Theorem 4.1.** *Let  $\mathcal{N}$  be a complete  $n$ -dimensional manifold without boundary and with bounded geometry. There exists an  $\epsilon_0 > 0$  small enough so that for all smooth  $(u_0, u_1)$  as in (1.6), (1.7) with*

$$\|(du_0, u_1)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} = \epsilon < \epsilon_0 \quad (4.3)$$

*there exists a unique, global smooth solution  $u : \mathbb{R} \times \mathbb{H}^4 \rightarrow \mathcal{N} \subset \mathbb{R}^N$  to (1.4) with initial data  $(u, \partial_t u)|_{t=0} = (u_0, u_1)$  and satisfying*

$$\sup_{t \in \mathbb{R}} \|(du(t), \partial_t u(t))\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \lesssim \epsilon. \quad (4.4)$$

*Moreover we have the following dispersive properties of the global wave map  $u$ : Let  $u(s, t, x)$  denote the harmonic map heat flow resolution of  $u(t, x)$ , let  $e$  be the caloric gauge with limiting frame  $e_\infty$ , and define  $\psi_s$  by  $e\psi_s = \partial_s u$ . Then,*

- (1) A priori control of dispersive norms:  $\psi_s$  satisfies the a priori estimates,

$$\begin{aligned} \|\psi_s\|_{S(\mathbb{R})} &\lesssim \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s|_{t=0}\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(L^2(\mathbb{H}^4; \mathbb{R}^n))} \\ &\lesssim \|(du_0, u_1)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} < \epsilon_0. \end{aligned} \quad (4.5)$$

- (2) Scattering: For each fixed  $s > 0$   $\vec{\psi}_s := (\psi_s, \partial_t \psi_s)$  scatters to free waves as  $t \rightarrow \pm\infty$ , that is, for each fixed  $s > 0$  there exist  $\mathbb{R}^n$ -valued free waves  $\phi_\pm(s)$  such that

$$\|\nabla_{t,x} \psi_s(s, t) - \nabla_{t,x} \phi_\pm(s, t)\|_{L^2(\mathbb{H}^4; \mathbb{R}^n)} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty. \quad (4.6)$$

- (3) Uniform control of higher derivatives: Higher regularity of the initial data  $\vec{\psi}_s(s, 0)$  is preserved by the flow. Moreover we have the following uniform estimates:

$$\|\Delta \psi_s\|_{S(\mathbb{R})} \lesssim \|(du_0, u_1)\|_{H^3 \times H^3(\mathbb{H}^4; T\mathcal{N})} \quad (4.7)$$

- (4) Pointwise decay: We have the following qualitative pointwise decay:

$$\|u(t, \cdot) - u_\infty\|_{L_x^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty. \quad (4.8)$$

We will now continue the proof of Theorem 4.1 that we began in Section 3.2. Indeed, our first main goal is still to establish the a priori estimates in Proposition 3.1. However, in order to prove Proposition 3.1 we will first deduce (4.5) under the bootstrap assumption (3.3).

To prove (4.5) we will study the Cauchy problem for the wave equation for  $\psi_s$  from Lemma 3.11. By expanding out the left-hand side of (3.96),  $\psi_s$  satisfies

$$\begin{aligned} \square_{\mathbb{H}^4} \psi_s &= \partial_s w - \eta^{\alpha\beta} F_{s\alpha} \psi_\beta - 2\eta^{\alpha\beta} A_\alpha \nabla_\beta \psi_s \\ &\quad - \eta^{\alpha\beta} A_\alpha A_\beta \psi_s - \eta^{\alpha\beta} (\nabla_\alpha A)_\beta \psi_s \end{aligned} \quad (4.9)$$

$$\vec{\psi}(s, 0) = (\psi_s(s, 0), \partial_t \psi_s(s, 0))$$

where  $w(s, t, x)$  solves (3.97). We remark that by Proposition 3.8 (in particular (3.88) restricted to  $\{t = 0\}$ ) and Proposition 3.5, the small data assumption (4.3) implies that

$$\|s^{\frac{1}{2}} \nabla_{t,x} \psi_s|_{t=0}\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(L^2(\mathbb{H}^4; \mathbb{R}^n))} \lesssim \|(du_0, u_1)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} \lesssim \epsilon_1. \quad (4.10)$$

Hence the proper setting for studying (4.9) is that of the Cauchy problem for initial data with *small critical Sobolev norm*. One of the key tools here is of course Strichartz estimates for the wave equation on  $\mathbb{H}^d$  established by Anker and Pierfelice [2]; see also [49, 50]. In what follows, we will refer to a triple  $(p, q, \gamma)$  as admissible if

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - \gamma, \quad \text{and} \quad \frac{1}{p} + \frac{d-1}{2q} \leq \frac{d-1}{4}, \quad \text{and} \quad p \geq 2, q \geq 2,$$

and if  $d = 3$  then  $(p, q, \gamma) \neq (2, \infty, 1)$ . Note that we also allow the energy estimates  $(p, q, \gamma) = (\infty, 2, 0)$ .

**Theorem 4.2.** [2, 49, 50] *Let  $J \subset \mathbb{R}$  be a time interval and let  $v : I \times \mathbb{H}^d \rightarrow \mathbb{R}$ , be a solution to the inhomogenous wave equation,*

$$\square_{\mathbb{H}^d} v = G, \quad \vec{v}(0) = (v_0, v_1) \quad (4.11)$$

and let  $(p, q, \gamma)$  be an admissible triple. Then,

$$\|\nabla_{t,x} v\|_{L_t^p(J; W^{-\gamma, q}(\mathbb{H}^d))} \lesssim \|(v_0, v_1)\|_{H^1 \times L^2(\mathbb{H}^d)} + \|G\|_{L_t^1(J; L_x^2(\mathbb{H}^d))} \quad (4.12)$$

*Remark 10.* We note that the Strichartz estimates above are a consequence of the dispersive estimates proved in [2, 50], which we will also use in the proof of the pointwise decay estimate (4.8) in Theorem 4.1.

*Proposition 4.3.* [2, 50] *Fix  $d \geq 3$ ,  $q \in (2, \infty)$ , and set  $\sigma = (d+1)(\frac{1}{2} - \frac{1}{q})$ .*

- *For  $0 < |t| \leq 2$ , the following short-time dispersive estimate holds:*

$$\|e^{\pm it\sqrt{-\Delta_{\mathbb{H}^d}}} f\|_{L^q(\mathbb{H}^d)} \lesssim_q |t|^{-(d-1)(\frac{1}{2} - \frac{1}{q})} \|f\|_{W^{\sigma, q'}(\mathbb{H}^d)} \quad (4.13)$$

- *For  $|t| \geq 2$ , the following long-time dispersive estimate holds:*

$$\|e^{\pm it\sqrt{-\Delta_{\mathbb{H}^d}}} f\|_{L^q(\mathbb{H}^d)} \lesssim_q |t|^{-\frac{3}{2}} \|f\|_{W^{\sigma, q'}(\mathbb{H}^d)} \quad (4.14)$$

We apply Theorem 4.2 to (4.9). Multiplying (4.9) by  $s^{\frac{1}{2}}$  and using (4.2) with  $d = 4$  and  $(p_1, q_1, \gamma_1) = (2, 8, 1)$  and  $(p_2, q_2, \gamma_2) = (\infty, 2, 0)$ , we deduce that for all  $s > 0$ ,

$$\begin{aligned} \|\psi_s(s)\|_{S_s(I)} &\lesssim \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s|_{t=0}\|_{L_x^2} + \|s^{\frac{1}{2}} \partial_s w\|_{L_t^1 L_x^2} + \|s^{\frac{1}{2}} F_s^\alpha \psi_\alpha\|_{L_t^1 L_x^2} \\ &\quad + \|s^{\frac{1}{2}} \nabla^\alpha A_\alpha \psi_s\|_{L_t^1 L_x^2} + \|s^{\frac{1}{2}} A^\alpha \nabla_\alpha \psi_s\|_{L_t^1 L_x^2} + \|s^{\frac{1}{2}} A^\alpha A_\alpha \psi_s\|_{L_t^1 L_x^2} \end{aligned} \quad (4.15)$$

Taking the  $L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2$  norm of both sides yields,

$$\begin{aligned} \|\psi_s\|_{S(I)} &\lesssim \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s|_{t=0}\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(L_x^2)} + \|s^{\frac{1}{2}} \partial_s w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(L_t^1 L_x^2)} \\ &\quad + \|s^{\frac{1}{2}} F_s^\alpha \psi_\alpha\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(L_t^1 L_x^2)} + \|s^{\frac{1}{2}} \nabla^\alpha A_\alpha \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(L_t^1 L_x^2)} \\ &\quad + \|s^{\frac{1}{2}} A^\alpha \nabla_\alpha \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(L_t^1 L_x^2)} + \|s^{\frac{1}{2}} A^\alpha A_\alpha \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(L_t^1 L_x^2)} \end{aligned} \quad (4.16)$$

The next order of business would be to control the terms on the right-hand side above, under an additional bootstrap assumption that

$$\|\psi_s\|_{S(I)} \leq 2C_1 \epsilon \quad (4.17)$$

for a large constant  $C_1$  to be determined later. As in the case of the first bootstrap constant  $C_0$ , in what follows we suppress the dependence of constants on  $C_1$ . This convention is justified by the fact that  $C_1$  is always accompanied by a factor of  $\epsilon$ .

Before directly addressing the terms on the right-hand side of (4.16) we first use (4.17) to establish a wider class of estimates. In particular, we use the fact that for each fixed  $t \in I$ ,  $\psi_s$  satisfies the *parabolic* equation (3.98) to deduce appropriately  $s$ -weighted estimates on the  $L_t^2 L_x^8$  and  $L_t^\infty L_x^2$  norms of higher derivatives of  $\psi_s$ . This is accomplished in the next section. The rough outline of the remainder of the paper is as follows:

- In Section 5 we use the parabolic equation satisfied by  $\psi_s$  to control higher derivatives of  $\psi_s$  and  $(-\Delta)^{-\frac{1}{2}} \partial_t \psi_s$  in  $L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8$  and of  $\nabla_{t,x} \psi_s$  in  $L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2$  (with appropriate  $s$ -weights) by  $\|\psi_s\|_{S(I)}$ . Here we directly invoke the  $L_x^p$  estimates for heat semi-group on  $\mathbb{H}^4$  from Lemma 2.11. We then establish a wider class of estimates for  $\psi_s$ ,  $\psi_\alpha$ , and  $A_\alpha$  that will be needed to close the a priori estimates (4.5) in the next section.
- In Section 6 we close the a priori estimates (4.5) under the bootstrap assumption (3.3). A significant part of the section is devoted to controlling the terms  $\|s^{\frac{1}{2}} \partial_s w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2}$  and  $\|s^{\frac{1}{2}} \nabla^\alpha A_\alpha \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2}$  from (4.16). Here we will make use of the parabolic equation (3.97) satisfied by  $w$ , again directly invoking properties of the heat semi-group from Lemma 2.11.
- Finally, in Section 7 we complete the proofs of Proposition 3.1 and Theorem 4.1.

## 5. PARABOLIC REGULARITY FOR $\psi_s$

In this section we establish  $L^p$ -type parabolic regularity estimates for  $\psi_s$  under the bootstrap assumption (4.17) by directly invoking properties of the heat semi-group  $e^{s\Delta}$  from Lemma 2.11. In Section 5.1 we then establish a wider collection of estimates for  $\psi_s$ ,  $\Psi$ , and  $A$  that will be needed in the sequel.

**Proposition 5.1.** *Let  $\psi_s$  be the heat tension field as in (3.94). Suppose that the bootstrap assumptions (3.3) and (4.17) hold. Then,*

$$\begin{aligned} & \|s \nabla \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} + \|s \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \lesssim \|\psi_s\|_{S(I)}, \\ & \|s^{\frac{3}{2}} \nabla^2 \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} + \|s^{\frac{3}{2}} \nabla \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \lesssim \|\psi_s\|_{S(I)}, \end{aligned} \quad (5.1)$$

where the implicit constants do not depend on  $C_0$  in (3.3) and  $C_1$  in (4.17).

The proof of Proposition 5.1 and the arguments that follow require the following lemma, which is a simple consequence of Proposition 3.5 and Lemma 3.9.

**Lemma 5.2.** *Assume that the bootstrap assumptions (3.3) and (4.17) hold. Then,*

$$\begin{aligned} & \|s^{\frac{1}{2}} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} + \|s \nabla \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} + \|s \mathbf{D} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} \lesssim \epsilon, \\ & \|s^{\frac{1}{2}} \Psi\|_{L_t^\infty L_{\frac{ds}{s}}^2 L_x^\infty} + \|s \nabla \Psi\|_{L_t^\infty L_{\frac{ds}{s}}^2 L_x^\infty} + \|s \mathbf{D} \Psi\|_{L_t^\infty L_{\frac{ds}{s}}^2 L_x^\infty} \lesssim \epsilon, \end{aligned} \quad (5.2)$$

and

$$\|s^{\frac{3}{8}} \mathbf{D} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^{\frac{16}{5}}} \simeq \|s^{\frac{3}{8}} \nabla \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^{\frac{16}{5}}} \lesssim \epsilon, \quad (5.3)$$

and

$$\begin{aligned} \|s^{\frac{1}{2}}A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} + \|s\nabla A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} + \|s^{\frac{3}{2}}\nabla^2 A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} &\lesssim \epsilon^2 \\ \|s\nabla_{t,x}\underline{A}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} + \|s^{\frac{3}{2}}\nabla\nabla_{t,x}\underline{A}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} &\lesssim \epsilon^2 \end{aligned} \quad (5.4)$$

*Remark 11.* By the torsion-free property  $\mathbf{D}_\alpha\Psi_\beta = \mathbf{D}_\beta\Psi_\alpha$ , the following analogue of (5.3) holds for  $\mathbf{D}_{t,x}\underline{\Psi}$ :

$$\|s^{\frac{3}{8}}\mathbf{D}_{t,x}\underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^{\frac{16}{5}}} \lesssim \epsilon \quad (5.3')$$

A similar remark applies to (5.2).

*Proof.* The bounds on the first terms in each line on the left in (5.2) are consequences of the Gagliardo-Nirenberg inequality,

$$\begin{aligned} \|s^{\frac{1}{2}}\Psi\|_{L_x^\infty}^2 &= \|s^{\frac{1}{2}}d_{t,x}u\|_{L_x^\infty}^2 \lesssim \|s^{\frac{1}{4}}(-\Delta)^{\frac{1}{2}}u\|_{L_x^8} \|s^{\frac{3}{4}}\nabla d_{t,x}u\|_{L_x^8} \\ &\lesssim \|s^{\frac{1}{4}}(-\Delta)^{\frac{5}{4}}u\|_{L_x^2} \|s^{\frac{1}{2}}\nabla^2 d_{t,x}u\|_{L_x^2}^{\frac{1}{2}} \|s\nabla^3 d_{t,x}u\|_{L_x^2}^{\frac{1}{2}} \end{aligned}$$

along with Proposition 3.5. The second terms on the left in (5.2) are controlled similarly. The third terms are then easy consequences of the previous bounds along with (5.4). To prove (5.4) we again use Gagliardo-Nirenberg to obtain

$$\|s^{\frac{1}{2}}A\|_{L_x^\infty} \lesssim s^{\frac{1}{2}}\|A\|_{L_x^8}^{\frac{1}{2}}\|\nabla A\|_{L_x^8}^{\frac{1}{2}} \lesssim \|A\|_{L_x^4}^{\frac{1}{4}}\|s^{\frac{1}{2}}\nabla^2 A\|_{L_x^2}^{\frac{1}{2}}\|s\nabla^3 A\|_{L_x^2}^{\frac{1}{4}},$$

and then conclude using Lemma 3.9. The terms involving  $s\nabla A$ ,  $s^{\frac{3}{2}}\nabla^2 A$ ,  $s\nabla_{t,x}\underline{A}$  and  $s^{\frac{3}{2}}\nabla\nabla_{t,x}\underline{A}$  in (5.4) are controlled in an identical fashion.

Finally, we turn to the proof of (5.3). The first equivalence follows from (3.89) (and taking  $\epsilon$  small enough), so it suffices to prove

$$\|s^{\frac{3}{8}}\nabla\Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^{\frac{16}{5}}} \lesssim \epsilon. \quad (5.5)$$

This estimate is an easy consequence of Proposition 3.5 and Gagliardo-Nirenberg,

$$\begin{aligned} \|s^{\frac{3}{8}}\nabla\Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^{\frac{16}{5}}} &\lesssim \|\nabla\Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^{\frac{1}{4}} \|\nabla^2\Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^{\frac{3}{4}} \\ &\lesssim \|\nabla d_{t,x}u\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^{\frac{1}{4}} \|\nabla^2 d_{t,x}u\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^{\frac{3}{4}} \lesssim \epsilon, \end{aligned}$$

which completes the proof.  $\square$

*Proof of Proposition 5.1.* Recall from Lemma 3.12 that  $\psi_s$  satisfies the heat equation

$$\partial_s\psi_s - \Delta\psi_s = F_s^b\psi_b + 2A^b\nabla_b\psi_s + A^bA_b\psi_s + \nabla^bA_b\psi_s =: G_1(s). \quad (5.6)$$

Therefore by Duhamel's formula we have

$$\psi_s(s) = e^{\frac{s}{2}\Delta}\psi_s(s/2) + \int_{\frac{s}{2}}^s e^{(s-s')\Delta}G_1(s')ds' \quad (5.7)$$

for each  $s > 0$ . Therefore,

$$s(-\Delta)^{\frac{1}{2}}\psi_s(s) = s^{\frac{1}{2}}(-\Delta)^{\frac{1}{2}}e^{\frac{s}{2}\Delta}s^{\frac{1}{2}}\psi_s(s/2) + s \int_{\frac{s}{2}}^s (-\Delta)^{\frac{1}{2}}e^{(s-s')\Delta}G_1(s')ds', \quad (5.8)$$



and

$$\begin{aligned} s(-\Delta)\psi_s(s) &= s^{\frac{1}{2}}(-\Delta)^{\frac{1}{2}}e^{\frac{s}{2}\Delta}s^{\frac{1}{2}}(-\Delta)^{\frac{1}{2}}\psi_s(s/2) \\ &\quad + s \int_{\frac{s}{2}}^s (-\Delta)^{\frac{1}{2}}e^{(s-s')\Delta}(-\Delta)^{\frac{1}{2}}G_1(s') ds'. \end{aligned} \quad (5.9)$$

Similarly, commuting  $\nabla_t$  through (5.7) and (5.8) we have

$$\begin{aligned} s\nabla_t\psi_s(s) &= s^{\frac{1}{2}}(-\Delta)^{\frac{1}{2}}e^{\frac{s}{2}\Delta}s^{\frac{1}{2}}(-\Delta)^{-\frac{1}{2}}\nabla_t\psi_s(s/2) \\ &\quad + s \int_{\frac{s}{2}}^s e^{(s-s')\Delta}\nabla_t G_1(s') ds' \end{aligned} \quad (5.10)$$

and

$$\begin{aligned} s(-\Delta)^{\frac{1}{2}}\nabla_t\psi_s(s) &= s^{\frac{1}{2}}(-\Delta)^{\frac{1}{2}}e^{\frac{s}{2}\Delta}s^{\frac{1}{2}}\nabla_t\psi_s(s/2) \\ &\quad + s \int_{\frac{s}{2}}^s (-\Delta)^{\frac{1}{2}}e^{(s-s')\Delta}\nabla_t G_1(s') ds' \end{aligned} \quad (5.11)$$

Next, from the heat semigroup bound (2.54) with  $p = 8$  applied to the right-hand side of (5.8) and the bound (2.53) applied to the right-hand side of (5.10) we see that

$$\begin{aligned} \|s\nabla_{t,x}\psi_s(s)\|_{L_t^2 L_x^8} &\lesssim \|(s/2)^{\frac{1}{2}}\psi_s(s/2)\|_{L_t^2 L_x^8} + \|(s/2)^{\frac{1}{2}}(-\Delta)^{-\frac{1}{2}}\nabla_t\psi_s(s/2)\|_{L_t^2 L_x^8} \\ &\quad + \int_{s/2}^s \frac{s}{(s-s')^{\frac{1}{2}}(s')^{\frac{1}{2}}} \|(s')^{\frac{3}{2}}G_1(s')\|_{L_t^2 L_x^8} \frac{ds'}{s'} \\ &\quad + \int_{s/2}^s \left(\frac{s}{s'}\right) \|(s')^2\nabla_t G_1(s')\|_{L_t^2 L_x^8} \frac{ds'}{s'} \end{aligned}$$

By Schur's test, it follows that

$$\begin{aligned} \|s\nabla_{t,x}\psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} &\lesssim \|\psi_s\|_{S(I)} + \|s^{\frac{3}{2}}G_1(s)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \\ &\quad + \|s^2\nabla_t G_1(s)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \end{aligned} \quad (5.12)$$

Similarly applying (2.54) with  $p = 2$  to (5.9) and (5.11) and arguing as above we obtain

$$\|s\nabla\nabla_{t,x}\psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^2} \lesssim \|\psi_s\|_{S(I)} + \|s^{\frac{3}{2}}\nabla_{t,x}G_1(s)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \quad (5.13)$$

Next, we show that

$$\begin{aligned} &\|s^{\frac{3}{2}}G_1\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} + \|s^2\nabla_t G_1(s)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} + \|s^{\frac{3}{2}}\nabla_{t,x}G_1\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \\ &\lesssim \epsilon^2 \left( \|s\nabla_{t,x}\psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} + \|s\nabla\nabla_{t,x}\psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \right) \\ &\quad + \epsilon^2 \|\psi_s\|_{S(I)} \end{aligned} \quad (5.14)$$

Note that (5.14) is sufficient to establish the first line in (5.1) since the terms inside the parenthesis on the right-hand side above can be absorbed into the left-hand side of either (5.12) or (5.13) as long as  $\epsilon$  is small enough.

We begin with the first  $L_t^2 L_x^8$  estimate of  $s^{\frac{3}{2}} G_1(s)$  in (5.14), bounding each of the terms in  $s^{\frac{3}{2}} G_1(s)$  as follows. Using (2.22) and (5.2) we have

$$\begin{aligned} \|s^{\frac{3}{2}} F_s^b \psi_b\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} &\lesssim \|s^{\frac{1}{2}} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \|s^{\frac{1}{2}} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty}^2 \\ &\lesssim \epsilon^2 \|\psi_s\|_{S(I)} \end{aligned}$$

Next, from (5.4) we see that

$$\begin{aligned} \|s^{\frac{3}{2}} A^b \nabla_b \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} &\lesssim \|s^{\frac{1}{2}} A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} \|s \nabla \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \\ &\lesssim \epsilon^2 \|s \nabla \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \end{aligned}$$

which can be absorbed onto the left-hand side of (5.12). Then note that

$$\begin{aligned} \|s^{\frac{3}{2}} A^b A_b \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} &\lesssim \|s^{\frac{1}{2}} A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty}^2 \|s^{\frac{1}{2}} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \\ &\lesssim \epsilon^4 \|\psi_s\|_{S(I)} \end{aligned} \quad (5.15)$$

Finally, using again (5.4) gives

$$\begin{aligned} \|s^{\frac{3}{2}} \nabla^b A_b \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} &\lesssim \|s \nabla A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} \|s^{\frac{1}{2}} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \\ &\lesssim \epsilon^2 \|\psi_s\|_{S(I)} \end{aligned} \quad (5.16)$$

Putting the previous four displayed equations together we have established that

$$\|s^{\frac{3}{2}} G_1\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \lesssim \epsilon^2 \|s \nabla \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} + \epsilon^2 \|\psi_s\|_{S(I)}$$

as desired.

Now we turn to the  $L_t^\infty L_x^2$  estimates of  $s^{\frac{3}{2}} \nabla_{t,x} G_1$  and the  $L_t^2 L_x^8$  estimate of  $s^2 \nabla_t G_1$  in (5.14). We claim that

$$\|s^{\frac{3}{2}} \nabla_{t,x} G_1\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \lesssim \epsilon^2 \|s \nabla \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} + \epsilon^2 \|\psi_s\|_{S(I)} \quad (5.17)$$

and

$$\|s^2 \nabla_t G_1\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} \lesssim \epsilon^2 \|s \nabla_t \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^8} + \epsilon^2 \|\psi_s\|_{S(I)} \quad (5.18)$$

We perform the details of the proof only for (5.17) since (5.18) can be proved similarly. We begin by computing  $\nabla_{t,x} G_1(s)$ . For any index  $\mu \in \{0, \dots, 4\}$  we have

$$\nabla_\mu G_1 = \nabla_\mu (F_s^b \psi_b) + 2 \nabla_\mu (A^b \nabla_b \psi_s) + \nabla_\mu (A^b A_b \psi_s) + \nabla_\mu (\nabla^b A_b \psi_s) \quad (5.19)$$

and we can expand and estimate each of the terms on the right above as follows. The first term is given by

$$\begin{aligned} \nabla_\mu (F_s^b \psi_b) &= \mathbf{D}_\mu (F_s^b \psi_b) - A_\mu F_s^b \psi_b = \mathbf{D}_\mu F_s^b \psi_b + F_s^b \mathbf{D}_b \psi_\mu - A_\mu F_s^b \psi_b \\ &= \mathbf{R}^{(1)}(\psi_\mu; \psi_s, \psi^b) \psi_b + \mathbf{R}^{(0)}(\mathbf{D}_\mu \psi_s, \psi^b) \psi_b + \mathbf{R}^{(0)}(\psi_s, \mathbf{D}_\mu \psi^b) \psi_b \\ &\quad + \mathbf{R}^{(0)}(\psi_s, \psi^b) \mathbf{D}_b \psi_\mu - A_\mu \mathbf{R}^{(0)}(\psi_s, \psi^b) \psi_b \end{aligned} \quad (5.20)$$

We then have

$$\begin{aligned} \|s^{\frac{3}{2}} \mathbf{R}^{(1)}(\Psi; \psi_s, \psi^b) \psi_b\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \\ \lesssim \|s^{\frac{1}{2}} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^4} \|s^{\frac{1}{2}} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty}^2 \|\Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4} \lesssim \epsilon^3 \|\psi_s\|_{S(I)} \end{aligned}$$

where the inequality follows from Sobolev embedding, (5.2), and (3.87). For the next term we use (3.88) and (5.2) to deduce that

$$\begin{aligned} \|s^{\frac{3}{2}} \mathbf{R}^{(0)}(\mathbf{D}_{t,x} \psi_s, \psi^b) \psi_b\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} &\lesssim \|s^{\frac{1}{2}} \mathbf{D}_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \|s^{\frac{1}{2}} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty}^2 \\ &\lesssim \epsilon^2 \|\psi_s\|_{S(I)} \end{aligned}$$

For the third term in (5.20) we use Sobolev embedding, (3.87) and (5.2) to conclude that

$$\begin{aligned} \|s^{\frac{3}{2}} \mathbf{R}^{(0)}(\psi_s, \mathbf{D}_{t,x} \psi^b) \psi_b\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \\ \lesssim \|s^{\frac{1}{2}} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^4} \|\Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4} \|s \mathbf{D}_{t,x} \underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} \lesssim \epsilon^2 \|\psi_s\|_{S(I)} \end{aligned}$$

The fourth term in (5.20) is controlled in an identical fashion. Finally for the fifth term we have

$$\begin{aligned} \|s^{\frac{3}{2}} A \mathbf{R}^{(0)}(\psi_s, \psi^b) \psi_b\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \\ \lesssim \|s^{\frac{1}{2}} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^4} \|A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4} \|s^{\frac{1}{2}} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty}^2 \lesssim \epsilon^3 \|\psi_s\|_{S(I)} \end{aligned}$$

using (3.89) and (5.2) in the last line.

The next term in (5.19) can be expanded as follows,

$$\begin{aligned} \nabla_\mu (A^b \nabla_b \psi_s) &= \nabla_\mu A^b \nabla_b \psi_s + A^b \nabla_\mu \nabla_b \psi_s \\ &= \nabla_\mu A^b \nabla_b \psi_s + A^b \nabla_b \nabla_\mu \psi_s \end{aligned}$$

where the commutation  $\nabla_\mu \nabla_b \psi_s = \nabla_b \nabla_\mu \psi_s$  does not pick up any domain curvature terms since the components  $\psi_s^j$  of  $\psi_s$  are treated in this context as scalar functions on  $\mathbb{R}^+ \times \mathcal{M}_I$ . We can then use (5.4) to show that

$$\begin{aligned} \|s^{\frac{3}{2}} \nabla_{t,x} (A^b \nabla_b \psi_s)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} &\lesssim \|s \nabla_{t,x} \underline{A}\|_{L_{\frac{ds}{s}}^\infty L_{t,x}^\infty} \|s^{\frac{1}{2}} \nabla \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \\ &\quad + \|s^{\frac{1}{2}} A\|_{L_{\frac{ds}{s}}^\infty L_{t,x}^\infty} \|s \nabla \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \\ &\lesssim \epsilon^2 \|\psi_s\|_{S(I)} + \epsilon^2 \|s \nabla \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \end{aligned}$$

Similarly,

$$\begin{aligned} \|s^{\frac{3}{2}} \nabla_{t,x} (A^b A_b \psi_s)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} &\lesssim \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \|s^{\frac{1}{2}} A\|_{L_{\frac{ds}{s}}^\infty L_{t,x}^\infty}^2 \\ &\quad + \|s^{\frac{1}{2}} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^4} \|A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4} \|s \nabla_{t,x} \underline{A}\|_{L_{\frac{ds}{s}}^\infty L_{t,x}^\infty} \lesssim \epsilon^2 \|\psi_s\|_{S(I)} \end{aligned}$$

For the last term we have

$$\nabla_\mu (\nabla^b A_b \psi_s) = \nabla_\mu \nabla^b A_b \psi_s + \nabla^b A_b \nabla_\mu \psi_s$$

If  $\mu = t$  note that we can commute  $\nabla_t \nabla^b A_b \psi_s = \nabla^b \nabla_t A_b \psi_s$ . Hence, using (3.91) and (5.4)

$$\begin{aligned} \|s^{\frac{3}{2}} \nabla_{t,x} (\nabla^b A_b \psi_s)\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} &\lesssim \|s \nabla \nabla_{t,x} \underline{A}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4} \|s^{\frac{1}{2}} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^4} \\ &\quad + \|s \nabla \underline{A}\|_{L_{\frac{ds}{s}}^\infty L_{t,x}^\infty} \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \\ &\lesssim \epsilon^2 \|\psi_s\|_{S(I)} \end{aligned}$$

This completes the proof of (5.17). The inequality (5.18) can be proved in a similar fashion, completing the proof of (5.14).

Finally, note that the second line in (5.1) can be proved in a nearly identical manner after applying  $(-\Delta)^{\frac{1}{2}}$  to (5.8), (5.10) and (5.9), (5.11). Indeed, arguing exactly as before, and this time noting that we have already established the first line in (5.1), we see that it suffices to prove that

$$\begin{aligned} & \|s^2 \nabla_{t,x} G_1\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^2 L_x^8} + \|s^2 \nabla \nabla_{t,x} G_1\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^2} \\ & \lesssim \epsilon \|s^{\frac{3}{2}} \nabla \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^2 L_x^8} + \epsilon \|s^{\frac{3}{2}} \nabla^2 \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^2} + \|\psi_s\|_{S(I)}, \end{aligned}$$

which in turn follows from a nearly identical argument to the one used to prove (5.14), this time requiring the full set of bounds in (5.2), (5.4).  $\square$

**5.1. A collection of estimates.** In the next section we will require a wider class of estimates for  $\psi_s, \Psi$ , and  $A$  than those that have already been explicitly stated. For convenience, we collect some of these estimates here.

The first simple observation is that we can use Proposition 3.5 and Proposition 5.1 together with the Gagliardo–Nirenberg inequality to establish the following estimates for  $\psi_s$ .

**Lemma 5.3.** *Let  $\psi_s$  be the heat tension field defined as in (3.94). Then,*

$$\begin{aligned} & \|s\psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^\infty} \\ & \lesssim \|s^{\frac{1}{2}} \nabla \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^2}^{\frac{1}{4}} \|s \nabla^2 \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^2}^{\frac{1}{2}} \|s^{\frac{3}{2}} \nabla^3 \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^2}^{\frac{1}{4}} \\ & \lesssim \|\psi_s\|_{S(I)}, \quad (5.21) \end{aligned}$$

$$\begin{aligned} & \|s^{\frac{5}{4}} \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^8} \\ & \lesssim \|s \nabla \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^2}^{\frac{1}{2}} \|s^{\frac{3}{2}} \nabla^2 \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^2}^{\frac{1}{2}} \lesssim \|\psi_s\|_{S(I)} \quad (5.22) \end{aligned}$$

and

$$\begin{aligned} & \|s^{\frac{3}{8}} s^{\frac{1}{2}} \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^{\frac{16}{5}}} \\ & \lesssim \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^2}^{\frac{1}{4}} \|s \nabla \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^\infty L_x^2}^{\frac{3}{4}} \lesssim \|\psi_s\|_{S(I)} \quad (5.23) \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \|s^{\frac{5}{8}} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^2 L_x^{16}} + \|s^{\frac{9}{8}} \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^2 L_x^{16}} \\ & \lesssim \|s^{\frac{1}{2}} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^2 L_x^8}^{\frac{3}{4}} \|s \nabla \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^2 L_x^8}^{\frac{1}{4}} \\ & \quad + \|s \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^2 L_x^8}^{\frac{3}{4}} \|s^{\frac{3}{2}} \nabla \nabla_{t,x} \psi_s\|_{L_{\frac{d_s}{s}}^\infty \cap L_{\frac{d_s}{s}}^2 L_t^2 L_x^8}^{\frac{1}{4}} \lesssim \|\psi_s\|_{S(I)}, \quad (5.24) \end{aligned}$$

*Proof.* The proof of (5.21) follows from two applications of the Gagliardo–Nirenberg inequality followed by (5.1). The proofs of (5.22) (5.23), and (5.24) are similar.  $\square$

With Lemma 5.3 in hand we can establish the following estimates for  $\Psi$  and  $A$ .

**Corollary 5.4.** *Let  $\Psi$  and  $A$  be as above and assume that the bootstrap assumptions (3.3) and (4.17) hold. Then,*

$$\|s^{\frac{1}{8}}\Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^{16}} \lesssim \|\psi_s\|_{S(I)} \quad (5.25)$$

$$\|s^{\frac{1}{4}}\Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^\infty} \lesssim \|\psi_s\|_{S(I)} \quad (5.26)$$

$$\|s^{\frac{1}{4}}\Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^8} \lesssim \|\psi_s\|_{S(I)} \quad (5.27)$$

$$\|s^{\frac{1}{2}}\mathbf{D}\Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^4} \lesssim \|\psi_s\|_{S(I)} \quad (5.28)$$

Moreover,

$$\|A\|_{L_{\frac{ds}{s}}^\infty L_t^1 L_x^\infty} \lesssim \|\psi_s\|_{S(I)}^2 \quad (5.29)$$

$$\|s^{\frac{1}{2}}A\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^\infty} \lesssim \epsilon \|\psi_s\|_{S(I)} \quad (5.30)$$

*Proof.* To prove (5.25) note that for each  $s > 0$  we have

$$\psi_\alpha(s) = - \int_s^\infty s' \nabla_\alpha \psi_s \frac{ds'}{s'} - \int_s^\infty s' A_\alpha \psi_s \frac{ds'}{s'}$$

Note that

$$\begin{aligned} \|s^{\frac{1}{8}}\Psi(s)\|_{L_t^2 L_x^{16}} &\lesssim \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{8}} \|(s')^{\frac{9}{8}} \nabla_{t,x} \psi_s\|_{L_t^2 L_x^{16}} \frac{ds'}{s'} \\ &\quad + \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{2}} \|(s')^{\frac{1}{8}} A\|_{L_{t,x}^\infty} \|(s')^{\frac{5}{8}} \psi_s\|_{L_t^2 L_x^{16}} \frac{ds'}{s'} \end{aligned}$$

Therefore, (5.25) follows by an application of Schur's test, (5.4) and (5.24).

Similarly, using Gagliardo-Nirenberg we obtain,

$$\begin{aligned} \|s^{\frac{1}{4}}\Psi(s)\|_{L_t^2 L_x^\infty} &\lesssim \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{4}} \|s' \nabla_{t,x} \psi_s\|_{L_t^2 L_x^8}^{\frac{1}{2}} \|(s')^{\frac{3}{2}} \nabla_{t,x} \psi_s\|_{L_t^2 L_x^8}^{\frac{1}{2}} \frac{ds'}{s'} \\ &\quad + \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{4}} \|(s')^{\frac{1}{2}} A\|_{L_{t,x}^\infty} \|(s')^{\frac{1}{2}} \psi_s\|_{L_t^2 L_x^8}^{\frac{1}{2}} \|s' \nabla_{t,x} \psi_s\|_{L_t^2 L_x^8}^{\frac{1}{2}} \frac{ds'}{s'} \end{aligned}$$

Then (5.26) follows by Schur's test as before.

Next, we prove (5.27). As above we have

$$\begin{aligned} \|s^{\frac{1}{4}}\Psi\|_{L_t^\infty L_x^8} &\lesssim \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{4}} \|(s')^{\frac{5}{4}} \nabla_{t,x} \psi_s\|_{L_t^\infty L_x^8} \frac{ds'}{s'} \\ &\quad + \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{4}} \|(s')^{\frac{1}{2}} A\|_{L_{t,x}^\infty} \|(s')^{\frac{3}{4}} \psi_s\|_{L_t^\infty L_x^8} \frac{ds'}{s'} \end{aligned}$$

By Schur's test followed by Sobolev embedding we see that

$$\begin{aligned} \|s^{\frac{1}{4}}\Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^8} &\lesssim \|s^{\frac{5}{4}} \nabla_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^8} \\ &\quad + \|s^{\frac{1}{2}} A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} \|s^{\frac{3}{4}} (-\Delta)^{\frac{3}{4}} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \\ &\lesssim \|\psi_s\|_{S(I)} \end{aligned}$$

where the last line above follows from (5.27), (5.4), interpolation, and (5.1).

To prove (5.28) we claim that it suffices to show

$$\|s^{\frac{1}{2}} \nabla \Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^4} \lesssim \|\psi_s\|_{S(I)} \quad (5.31)$$

To see this, note that using (5.27), Gagliardo-Nirenberg, (3.89), and (3.91) we have

$$\begin{aligned} & \left| \|s^{\frac{1}{2}} \nabla \Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^4} - \|s^{\frac{1}{2}} \mathbf{D} \Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^4} \right| \\ & \lesssim \|A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4}^{\frac{1}{2}} \|s^{\frac{1}{2}} \nabla A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4}^{\frac{1}{2}} \|s^{\frac{1}{4}} \Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^8} \lesssim \epsilon \|\psi_s\|_{S(I)} \end{aligned}$$

To prove (5.31), we start with

$$\nabla_b \psi_\alpha = - \int_s^\infty \nabla_b \mathbf{D}_\alpha \psi_s \, ds'$$

It follows that

$$\begin{aligned} \|s^{\frac{1}{2}} \nabla \Psi\|_{L_t^\infty L_x^4} & \lesssim \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{2}} \|(s')^{\frac{3}{2}} \nabla \nabla_{t,x} \psi_s\|_{L_t^\infty L_x^4} \frac{ds'}{s'} \\ & \quad + \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{2}} \|(s')^{\frac{1}{2}} A\|_{L_t^\infty L_x^\infty} \|s' \nabla_{t,x} \psi_s\|_{L_t^\infty L_x^4} \frac{ds'}{s'} \\ & \quad + \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{2}} \|s' \nabla A\|_{L_t^\infty L_x^\infty} \|(s')^{\frac{1}{2}} \psi_s\|_{L_t^\infty L_x^4} \frac{ds'}{s'} \end{aligned}$$

By Schur's test and (5.4) we obtain (5.31).

Next, using (3.83) and (2.22) we see that

$$\begin{aligned} \|A\|_{L_{\frac{ds}{s}}^\infty L_t^1 L_x^\infty} & \lesssim \int_0^\infty \|s^{\frac{3}{4}} \psi_s\|_{L_t^2 L_x^\infty} \|s^{\frac{1}{4}} \Psi\|_{L_t^2 L_x^\infty} \frac{ds}{s} \\ & \lesssim \left( \int_0^\infty \|s^{\frac{3}{4}} \psi_s\|_{L_t^2 L_x^\infty}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \left( \int_0^\infty \|s^{\frac{1}{4}} \Psi\|_{L_t^2 L_x^\infty}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\ & \lesssim \left( \int_0^\infty \|s^{\frac{1}{2}} \psi_s\|_{L_t^2 L_x^8} \|s \nabla \psi_s\|_{L_t^2 L_x^8} \frac{ds}{s} \right)^{\frac{1}{2}} \left( \int_0^\infty \|s^{\frac{1}{4}} \Psi\|_{L_t^2 L_x^\infty}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\ & \lesssim \|\psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 S_s(I)}^2 = \|\psi_s\|_{S(I)}^2 \end{aligned}$$

where we used (5.26) in the last line. Lastly, we establish (5.30). Arguing similarly, we have

$$\|s^{\frac{1}{2}} A\|_{L_t^\infty L_x^\infty} \lesssim \int_s^\infty \left(\frac{s}{s'}\right)^{\frac{1}{2}} \|s' \psi_s\|_{L_t^\infty L_x^\infty} \|(s')^{\frac{1}{2}} \Psi\|_{L_t^\infty L_x^\infty} \frac{ds'}{s'}$$

Using Schur's test, we deduce that

$$\begin{aligned} \|s^{\frac{1}{2}} A\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^\infty} & \lesssim \|s \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^\infty} \|s^{\frac{1}{2}} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} \\ & \lesssim \epsilon \|\psi_s\|_{S(I)} \end{aligned}$$

where the last line follows from (5.21) and (5.2). This completes the proof.  $\square$

6. ANALYSIS OF THE WAVE EQUATION SATISFIED BY  $\psi_s$ 

In this section we complete the proof of (4.5) under the bootstrap assumption (3.3). Indeed we establish the following proposition.

**Proposition 6.1.** *Let  $\psi_s$  denote the heat tension field as defined in (3.94). There exists  $\epsilon_2 > 0$  small enough so that if the bootstrap assumptions (3.3) and (4.17) hold with  $\epsilon < \epsilon_2$ , then*

$$\begin{aligned} \|\psi_s\|_{\mathcal{S}(I)} &\lesssim \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s|_{t=0}\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_x^2} \\ &\lesssim \|(du_0, u_1)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} = \epsilon \end{aligned} \quad (6.1)$$

for an implicit constant that is independent of  $I$ , as well as the constants  $C_0$  from (3.3) and  $C_1$  from (4.17).

*Remark 12.* Note that Proposition 6.1 closes the bootstrap hypothesis (4.17). The other bootstrap assumption (3.3) will be closed in Section 7. We also remark that the second inequality in (6.1) has already been established using the theory of the harmonic map heat flow and our assumptions on the initial data  $(u_0, u_1)$ .

Once we have established the a priori estimates in Proposition 3.1, the estimate (6.1) holds for the heat tension field  $\psi_s$  associated to any smooth wave map  $u$  on any time interval  $I$  and with smooth data as in (3.1). One can then deduce the following uniform control of the  $\mathcal{S}(I)$  norm of  $\Delta\psi_s$ . This control of higher derivatives of  $\psi_s$  will be used in the proof of Theorem 4.1.

**Corollary 6.2.** *Let  $(u_0, u_1)$  be initial data as in (1.6), (1.7) with*

$$\|(du_0, u_1)\|_{H^3 \times H^3(\mathbb{H}^4; T\mathcal{N})} =: B_0 < \infty \quad (6.2)$$

and let  $(u, \partial_t u)$  be a smooth wave map on a time interval  $I \subset \mathbb{R}$  with initial data  $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ . Let  $\psi_s$  denote the heat tension field associated to  $u$  as defined above. Then, there exists  $\epsilon_3 > 0$  small enough so that if

$$\|\psi_s\|_{\mathcal{S}(I)} \leq \epsilon_3 \quad (6.3)$$

we have the following uniform estimates on the  $\mathcal{S}(I)$  norm of  $(\Delta\psi_s, \partial_t \Delta\psi_s)$  in terms of the  $H^3 \times H^3$  norm of the data  $(du_0, u_1)$ ,

$$\|\Delta\psi_s\|_{\mathcal{S}(I)} \lesssim B_0 \quad (6.4)$$

*Remark 13.* We note that given the a priori control (6.3), the uniform control of higher derivatives of  $\Delta\psi_s$  as in Corollary 6.2 follows from essentially the same proof as the proof of Proposition 6.1 after taking the Laplacian of (4.9). Indeed, one obtains an effectively linear equation for  $\Delta\psi_s$  with coefficients that are small because of (6.3). To avoid repetition, we omit the details of the proof of Corollary 6.2 and instead refer the reader to the details of the proof of Proposition 6.1.

Lastly, we remark that the introduction of  $\epsilon_3$  can be avoided by carefully making use of divisible norms; cf. Remark 7.

The proof of Proposition 6.1 requires the following two additional lemmas.

**Lemma 6.3.** *Let  $w = w(s, t, x)$  denote the wave tension field defined as in (3.95). Then, assuming (3.3) and (4.17), we have*

$$\|s^{\frac{1}{2}}\partial_s w\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} \lesssim \epsilon \|\psi_s\|_{S(I)}^2 \quad (6.5)$$

$$\|s^{\frac{1}{4}}\nabla w\|_{L^2_{\frac{d_s}{s}} L^2_t L^2_x} \lesssim \epsilon \|\psi_s\|_{S(I)}^2 \quad (6.6)$$

**Corollary 6.4.** *Under the assumptions of the previous lemma, we have*

$$\|\nabla^\alpha A_\alpha\|_{L^{\infty}_{\frac{d_s}{s}} L^2_t L^{\frac{8}{3}}_x} \lesssim \|\psi_s\|_{S(I)}^2 \quad (6.7)$$

We postpone the proofs of Lemma 6.3 and Corollary 6.4 and complete the proof of Proposition 6.1.

*Proof of Proposition 6.1.* The starting point is (4.16) which we recall was the inequality

$$\begin{aligned} \|\psi_s\|_{S(I)} &\lesssim \|s^{\frac{1}{2}}\nabla_{t,x}\psi_s|_{t=0}\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^2_x} + \|s^{\frac{1}{2}}\partial_s w\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} \\ &\quad + \|s^{\frac{1}{2}}F_s^\alpha \psi_\alpha\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} + \|s^{\frac{1}{2}}\nabla^\alpha A_\alpha \psi_s\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} \\ &\quad + \|s^{\frac{1}{2}}A^\alpha \nabla_\alpha \psi_s\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} + \|s^{\frac{1}{2}}A^\alpha A_\alpha \psi_s\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} \end{aligned}$$

established in Section 4 using Strichartz estimates.

We estimate the last five terms on the right-hand side above as follows. First by Lemma 6.3 we have

$$\|s^{\frac{1}{2}}\partial_s w\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} \lesssim \epsilon \|\psi_s\|_{S(I)}^2$$

Next using Sobolev embedding, (5.25) and (3.44) we have

$$\begin{aligned} \|s^{\frac{1}{2}}F_s^\alpha \psi_\alpha\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} &\lesssim \|s^{\frac{1}{8}}\Psi\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^2_t L^{16}_x}^2 \|s^{\frac{1}{4}}\partial_s u\|_{L^{\infty}_{\frac{d_s}{s}} L^{\infty}_t L^{\frac{8}{3}}_x} \\ &\lesssim \|s^{\frac{1}{8}}\Psi\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^2_t L^{16}_x}^2 \|s^{\frac{1}{4}}(-\Delta)^{\frac{1}{4}}\partial_s u\|_{L^{\infty}_{\frac{d_s}{s}} L^{\infty}_t L^2_x} \\ &\lesssim \epsilon \|\psi_s\|_{S(I)}^2 \end{aligned}$$

For the next term we use (6.7) to deduce

$$\begin{aligned} \|s^{\frac{1}{2}}\nabla^\alpha A_\alpha \psi_s\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} &\lesssim \|\nabla^\alpha A_\alpha\|_{L^{\infty}_{\frac{d_s}{s}} L^2_t L^{\frac{8}{3}}_x} \|s^{\frac{1}{2}}\psi_s\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} \\ &\lesssim \epsilon \|\psi_s\|_{S(I)}^2 \end{aligned}$$

Similarly, this time using (5.29) we see that

$$\begin{aligned} \|s^{\frac{1}{2}}A^\alpha \nabla_\alpha \psi_s\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} &\lesssim \|A\|_{L^{\infty}_{\frac{d_s}{s}} L^1_t L^{\infty}_x} \|s^{\frac{1}{2}}\nabla_{t,x}\psi_s\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^{\infty}_t L^2_x} \\ &\lesssim \epsilon \|\psi_s\|_{S(I)}^2 \end{aligned}$$

Finally, we use (3.89), (5.29), and Sobolev embedding to estimate

$$\begin{aligned} \|s^{\frac{1}{2}}A^\alpha A_\alpha \psi_s\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^1_t L^2_x} &\lesssim \|A\|_{L^{\infty}_{\frac{d_s}{s}} L^{\infty}_t L^4_x} \|A\|_{L^{\infty}_{\frac{d_s}{s}} L^1_t L^{\infty}_x} \|s^{\frac{1}{2}}\psi_s\|_{L^{\infty}_{\frac{d_s}{s}} \cap L^2_{\frac{d_s}{s}} L^{\infty}_t L^4_x} \\ &\lesssim \epsilon^3 \|\psi_s\|_{S(I)}^2 \end{aligned}$$



Putting this together we have proved that

$$\|\psi_s\|_{S(I)} \lesssim \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s \upharpoonright_{t=0}\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_x^2} + \epsilon \|\psi_s\|_{S(I)}^2 \quad (6.8)$$

By the usual continuity argument it follows that

$$\|\psi_s\|_{S(I)} \lesssim \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s \upharpoonright_{t=0}\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_x^2}$$

as desired. In view of Remark 12 this completes the proof of Proposition 6.1.  $\square$

It remains to prove Lemma 6.3 and Corollary 6.4.

*Proof of Lemma 6.3.* We begin with the proof of (6.5). We first show that

$$\|s^{-\frac{1}{2}} w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} + \|s^{\frac{1}{2}} \Delta w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \lesssim \epsilon \|\psi_s\|_{S(I)}^2 \quad (6.9)$$

Recall that  $w$  obeys the covariant heat equation (3.97). Expanding out the right-hand side using (2.22) and (2.24) gives

$$\begin{aligned} \partial_s w - \mathbf{D}^a \mathbf{D}_a w &= \mathbf{R}^{(0)}(w, \psi^a) \psi_a + 3\mathbf{R}^{(0)}(\psi_\mu, \psi^a) \mathbf{D}^\mu \psi_a + \mathbf{R}^{(1)}(\psi^\mu; \psi_\mu, \psi^a) \psi_a \\ &\quad + \mathbf{R}^{(1)}(\psi^a; \psi^\mu, \psi_a) \psi_\mu + \mathbf{R}^{(0)}(\psi_\mu, \mathbf{D}^\mu \psi^a) \psi_a + \mathbf{R}^{(0)}(\mathbf{D}^a \psi^\mu, \psi_a) \psi_\mu \end{aligned}$$

On the other hand, expanding out the left-hand side using  $\mathbf{D}_a = \nabla_a + A_a$  yields,

$$\begin{aligned} \partial_s w - \Delta w &= -2A^a \partial_a w - \nabla^a A_a w - A^a A_a w \\ &\quad + \mathbf{R}^{(0)}(w, \psi^a) \psi_a + 3\mathbf{R}^{(0)}(\psi_\mu, \psi^a) \mathbf{D}_a \psi^\mu + \mathbf{R}^{(1)}(\psi^\mu; \psi_\mu, \psi^a) \psi_a \\ &\quad + \mathbf{R}^{(1)}(\psi^a; \psi^\mu, \psi_a) \psi_\mu + \mathbf{R}^{(0)}(\psi_\mu, \mathbf{D}^a \psi^\mu) \psi_a + \mathbf{R}^{(0)}(\mathbf{D}^a \psi^\mu, \psi_a) \psi_\mu \\ &=: G_2(s) \end{aligned} \quad (6.10)$$

Now, since  $w(s, t, x)$  satisfies  $w(0, t, x) = 0$  we can write

$$s^{-\frac{1}{2}} w(s) = \int_0^{\frac{s}{2}} s^{-\frac{1}{2}} e^{(s-s')\Delta} G_2(s') ds' + \int_{\frac{s}{2}}^s s^{-\frac{1}{2}} e^{(s-s')\Delta} G_2(s') ds' \quad (6.11)$$

It follows that

$$s^{\frac{1}{2}} \Delta w = \int_0^{\frac{s}{2}} s^{\frac{1}{2}} \Delta e^{(s-s')\Delta} G_2(s') ds' - \int_{\frac{s}{2}}^s s^{\frac{1}{2}} (-\Delta)^{\frac{1}{2}} e^{(s-s')\Delta} (-\Delta)^{\frac{1}{2}} G_2(s') ds' \quad (6.12)$$

Next, we use (2.53) on (6.11) and then (2.55) and (2.54) on (6.12) to deduce that

$$\begin{aligned} \|s^{-\frac{1}{2}} w\|_{L_t^1 L_x^2} &\lesssim \int_0^{\frac{s}{2}} \left(\frac{s'}{s}\right)^{\frac{1}{2}} \|(s')^{\frac{1}{2}} G_2(s')\|_{L_t^1 L_x^2} \frac{ds'}{s'} \\ &\quad + \int_{\frac{s}{2}}^s \left(\frac{s'}{s}\right)^{\frac{1}{2}} \|(s')^{\frac{1}{2}} G_2(s')\|_{L_t^1 L_x^2} \frac{ds'}{s'} \end{aligned}$$

and

$$\begin{aligned}
\|s^{\frac{1}{2}}\Delta w\|_{L_t^1 L_x^2} &\lesssim \int_0^{\frac{s}{2}} \frac{s^{\frac{1}{2}}(s')^{\frac{1}{2}}}{s-s'} \|(s')^{\frac{1}{2}}G_2(s')\|_{L_t^1 L_x^2} \frac{ds'}{s'} \\
&\quad + \int_{\frac{s}{2}}^s \frac{s^{\frac{1}{2}}}{(s-s')^{\frac{1}{2}}} \|s'\nabla G_2(s')\|_{L_t^1 L_x^2} \frac{ds'}{s'} \\
&\lesssim \int_0^{\frac{s}{2}} \left(\frac{s'}{s}\right)^{\frac{1}{2}} \|(s')^{\frac{1}{2}}G_2(s')\|_{L_t^1 L_x^2} \frac{ds'}{s'} \\
&\quad + \int_{\frac{s}{2}}^s \frac{s^{\frac{1}{2}}}{(s-s')^{\frac{1}{2}}} \|s'\nabla G_2(s')\|_{L_t^1 L_x^2} \frac{ds'}{s'}
\end{aligned}$$

Applying Schur's test to the two previous displayed inequalities we obtain

$$\begin{aligned}
\|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} + \|s^{\frac{1}{2}}\Delta w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} &\lesssim \|s^{\frac{1}{2}}G_2\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \\
&\quad + \|s\nabla G_2\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2}
\end{aligned}$$

Hence it suffices to show that

$$\begin{aligned}
\|s^{\frac{1}{2}}G_2\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} &\lesssim \epsilon^2 \left( \|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} + \|s^{\frac{1}{2}}\Delta w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \right) \\
&\quad + \epsilon \|\psi_s\|_{\mathcal{S}(I)}^2
\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
\|s\nabla G_2\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} &\lesssim \epsilon^2 \left( \|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} + \|s^{\frac{1}{2}}\Delta w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \right) \\
&\quad + \epsilon \|\psi_s\|_{\mathcal{S}(I)}^2
\end{aligned} \tag{6.14}$$

We first estimate each of the terms that arise on the left-hand side of (6.13), recalling that  $G_2(s)$  is defined in (6.10). First, using (5.4) we have

$$\begin{aligned}
\|s^{\frac{1}{2}}A^a\partial_a w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} &\lesssim \|s^{\frac{1}{2}}A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} \|\nabla w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \\
&\lesssim \epsilon^2 \|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2}^{\frac{1}{2}} \|s^{\frac{1}{2}}\Delta w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2}^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\|s^{\frac{1}{2}}\nabla^a A_a w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} &\lesssim \|s\nabla A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} \|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \\
&\lesssim \epsilon^2 \|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2}
\end{aligned}$$

and

$$\begin{aligned}
\|s^{\frac{1}{2}}A^a A_a w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} &\lesssim \|s^{\frac{1}{2}}A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty}^2 \|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \\
&\lesssim \epsilon^4 \|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2}
\end{aligned}$$

Next, note that

$$\begin{aligned}
\|s^{\frac{1}{2}}\mathbf{R}^{(0)}(w, \psi^a)\psi_a\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} &\lesssim \|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \|s^{\frac{1}{2}}\Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty}^2 \\
&\lesssim \epsilon^2 \|s^{-\frac{1}{2}}w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2}
\end{aligned}$$

where we have used (5.2) in the last line above.

Next, we observe that using (3.89), (5.25), Sobolev embedding, and Gagliardo–Nirenberg we have

$$\begin{aligned}
& \|s^{\frac{1}{2}} \mathbf{R}^{(0)}(\psi_\mu, \psi^a) \mathbf{D}_a \psi^\mu\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \\
& \lesssim \|s^{\frac{1}{8}} \Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^{16}}^2 \left( \|s^{\frac{1}{4}} \nabla \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^{\frac{8}{3}}} + \|s^{\frac{1}{4}} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^8} \|A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4} \right) \\
& \lesssim \|\psi_s\|_{S(I)}^2 \|\nabla \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^{\frac{1}{2}} \|s^{\frac{1}{2}} \nabla^2 \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^{\frac{1}{2}} \lesssim \epsilon \|\psi_s\|_{S(I)}^2
\end{aligned}$$

where in the last line above we used (3.93). Arguing similarly we have

$$\begin{aligned}
& \|s^{\frac{1}{2}} \mathbf{R}^{(1)}(\psi^\mu; \psi_\mu, \psi^a) \psi_a\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \\
& \lesssim \|s^{\frac{1}{8}} \Psi\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^2 L_x^{16}}^2 \|s^{\frac{1}{4}} \Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^8} \|\Psi\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4} \lesssim \epsilon^2 \|\psi_s\|_{S(I)}^2
\end{aligned}$$

The remaining terms in (6.10) are controlled in a similar fashion as the previous two, thus completing the proof of (6.13). A nearly identical argument is used to prove (6.14). This proves (6.9). Finally, to complete the proof of (6.5) we note that from equation (6.10) we have

$$\begin{aligned}
& \|s^{\frac{1}{2}} \partial_s w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \lesssim \|s^{\frac{1}{2}} \Delta w\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} + \|s^{\frac{1}{2}} G_2\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^1 L_x^2} \\
& \lesssim \epsilon \|\psi_s\|_{S(I)}^2
\end{aligned}$$

where we have used (6.13) followed by (6.9) in the last line above.

Next, we establish (6.6). First note that it suffices to show that

$$\|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w\|_{L_{\frac{ds}{s}}^2 L_t^2 L_x^2} \lesssim \epsilon \|\psi_s\|_{S(I)}^2 \quad (6.15)$$

Using that  $w(0, t, x) = 0$  from the Duhamel formula and with  $G_2$  as above we have

$$s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w(s) = \int_0^s s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} e^{(s-s')\Delta} G_2(s') ds'$$

Using (2.54) we see that

$$\|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w(s)\|_{L_t^2 L_x^2} \lesssim \int_0^s \frac{s^{\frac{1}{4}} (s')^{\frac{1}{4}}}{(s-s')^{\frac{1}{2}}} \|(s')^{\frac{3}{4}} G_2(s')\|_{L_t^2 L_x^2} \frac{ds'}{s'}$$

By Schur's test we then deduce that

$$\|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w(s)\|_{L_{\frac{ds}{s}}^2 L_t^2 L_x^2} \lesssim \|s^{\frac{3}{4}} G_2(s)\|_{L_{\frac{ds}{s}}^2 L_t^2 L_x^2}$$

Therefore, it suffices to show that

$$\|s^{\frac{3}{4}} G_2(s)\|_{L_{\frac{ds}{s}}^2 L_t^2 L_x^2} \lesssim \epsilon^2 \|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w\|_{L_{\frac{ds}{s}}^2 L_t^2 L_x^2} + \epsilon \|\psi_s\|_{S(I)}^2 \quad (6.16)$$

We estimate each of the terms in  $G_2$ , which is defined in (6.10). We first control all of the terms involving  $w$ . Using (5.30) we have

$$\begin{aligned}
& \|s^{\frac{3}{4}} A^a \partial_a w\|_{L_{\frac{ds}{s}}^2 L_t^2 L_x^2} \lesssim \|s^{\frac{1}{2}} A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^\infty} \|s^{\frac{1}{4}} \nabla w\|_{L_{\frac{ds}{s}}^2 L_t^2 L_x^2} \\
& \lesssim \epsilon^2 \|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w\|_{L_{\frac{ds}{s}}^2 L_t^2 L_x^2}
\end{aligned}$$

Next, we use (3.91) to deduce that

$$\begin{aligned} \|s^{\frac{3}{4}} \nabla^a A_a w\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} &\lesssim \|s^{\frac{1}{2}} \nabla A\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^4_x} \|s^{\frac{1}{4}} w\|_{L^2_{\frac{ds}{s}} L^2_t L^4_x} \\ &\lesssim \|s^{\frac{1}{2}} \nabla^2 A\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^2_x} \|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} \\ &\lesssim \epsilon^2 \|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} \end{aligned}$$

Continuing as above, using (5.30) and (3.89) we observe

$$\begin{aligned} \|A^a A_a w\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} &\lesssim \|s^{\frac{1}{2}} A\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^\infty_x} \|A\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^4_x} \|s^{\frac{1}{4}} w\|_{L^2_{\frac{ds}{s}} L^2_t L^4_x} \\ &\lesssim \epsilon^4 \|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} \end{aligned}$$

And lastly, using (5.27) and (4.17) we see that

$$\begin{aligned} \|\mathbf{R}^{(0)}(w, \psi^a) \psi_a\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} &\lesssim \|s^{\frac{1}{4}} w\|_{L^2_{\frac{ds}{s}} L^2_t L^4_x} \|s^{\frac{1}{4}} \Psi\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^8_x}^2 \\ &\lesssim \|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} \|\psi_s\|_{S(I)}^2 \\ &\lesssim \epsilon^2 \|s^{\frac{1}{4}} (-\Delta)^{\frac{1}{2}} w\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} \end{aligned}$$

Next, we estimate the terms that do not contain  $w$ . To start we use (5.25), (5.27), and (5.3) to establish that

$$\begin{aligned} \|s^{\frac{3}{4}} \mathbf{R}^{(0)}(\psi_\mu, \psi^a) \mathbf{D}_a \psi^\mu\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} &\lesssim \|s^{\frac{1}{8}} \Psi\|_{L^2_{\frac{ds}{s}} L^2_t L^{16}_x} \|s^{\frac{1}{4}} \Psi\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^8_x} \|s^{\frac{3}{8}} \mathbf{D} \Psi\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^{16}_x} \lesssim \epsilon \|\psi_s\|_{S(I)}^2 \end{aligned}$$

Similarly, we use (5.25), (5.27), Sobolev embedding, (5.3), and (3.87) to show that

$$\begin{aligned} \|s^{\frac{3}{4}} \mathbf{R}^{(1)}(\psi^\mu; \psi_\mu, \psi^a) \psi_a\|_{L^2_{\frac{ds}{s}} L^2_t L^2_x} &\lesssim \|s^{\frac{1}{8}} \Psi\|_{L^2_{\frac{ds}{s}} L^2_t L^{16}_x} \|s^{\frac{1}{4}} \Psi\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^8_x} \|s^{\frac{3}{8}} \Psi\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^{16}_x} \|\Psi\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^4_x} \\ &\lesssim \|\psi_s\|_{S(I)}^2 \|s^{\frac{3}{8}} \nabla \Psi\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^{16}_x} \|\nabla \Psi\|_{L^\infty_{\frac{ds}{s}} L^\infty_t L^2_x} \\ &\lesssim \epsilon^2 \|\psi_s\|_{S(I)}^2 \end{aligned}$$

The remaining terms in (6.10) can be handled in an identical fashion as the previous two, concluding the proof of (6.16). This completes the proof of (6.6).  $\square$

Lastly, we prove Corollary 6.4.

*Proof of Corollary 6.4.* Recall that

$$A_\alpha(s) = - \int_s^\infty s' F_{s\alpha} \frac{ds'}{s'}$$

It follows that

$$\nabla^\alpha A_\alpha(s) = - \int_s^\infty s' \mathbf{D}^\alpha F_{s\alpha} \frac{ds'}{s'} + \int_s^\infty s' [A^\alpha, F_{s\alpha}] \frac{ds'}{s'}$$

and thus,

$$\|\nabla^\alpha A_\alpha\|_{L_t^\infty L_t^2 L_x^{\frac{8}{3}}} \lesssim \int_0^\infty \|s' \mathbf{D}^\alpha F_{s\alpha}\|_{L_t^2 L_x^{\frac{8}{3}}} \frac{ds'}{s'} + \int_0^\infty \|s' [A^\alpha, F_{s\alpha}]\|_{L_t^2 L_x^{\frac{8}{3}}} \frac{ds'}{s'} \quad (6.17)$$

Using the notation from Section 2.4 and recalling that  $w = \mathbf{D}^\alpha \psi_\alpha$  we write

$$\mathbf{D}^\alpha F_{s\alpha} = \mathbf{R}^{(1)}(\psi^\alpha; \psi_s, \psi_\alpha) + \mathbf{R}^{(0)}(\mathbf{D}^\alpha \psi_s, \psi_\alpha) + \mathbf{R}^{(0)}(\psi_s, w)$$

and we estimate each term on the right as follows:

$$\|s' \mathbf{R}^{(1)}(\psi^\alpha; \psi_s, \psi_\alpha)\|_{L_t^2 L_x^{\frac{8}{3}}} \lesssim \|(s')^{\frac{1}{2}} \psi_s\|_{L_t^2 L_x^8} \|(s')^{\frac{1}{4}} \Psi\|_{L_t^\infty L_x^8}^2$$

Next,

$$\begin{aligned} \|\mathbf{R}^{(0)}(\mathbf{D}^\alpha \psi_s, \psi_\alpha)\|_{L_t^2 L_x^{\frac{8}{3}}} &\lesssim \|(s')^{\frac{7}{8}} \mathbf{D}_{t,x} \psi_s\|_{L_t^\infty L_x^{\frac{16}{5}}} \|(s')^{\frac{1}{8}} \Psi\|_{L_t^2 L_x^{16}} \\ &\lesssim \|(s')^{\frac{7}{8}} \nabla_{t,x} \psi_s\|_{L_t^\infty L_x^{\frac{16}{5}}} \|(s')^{\frac{1}{8}} \Psi\|_{L_t^2 L_x^{16}} \end{aligned}$$

where the last line follows using (3.89). For the term involving  $w$  we have

$$\begin{aligned} \|\mathbf{R}^{(0)}(\psi_s, w)\|_{L_t^2 L_x^{\frac{8}{3}}} &\lesssim \|(s')^{\frac{3}{4}} \psi_s\|_{L_t^\infty L_x^8} \|(s')^{\frac{1}{4}} w\|_{L_t^2 L_x^4} \\ &\lesssim \|(s')^{\frac{1}{2}} \psi_s\|_{L_t^\infty L_x^4} \|s' \nabla \psi_s\|_{L_t^\infty L_x^4}^{\frac{1}{2}} \|(s')^{\frac{1}{4}} \nabla w\|_{L_t^2 L_x^2} \\ &\lesssim \|(s')^{\frac{1}{2}} \nabla \psi_s\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|s' \nabla^2 \psi_s\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|(s')^{\frac{1}{4}} \nabla w\|_{L_t^2 L_x^2} \end{aligned}$$

For the last term on the right-hand side of (6.17) we will use the estimate

$$\begin{aligned} \|s' [A^\alpha, F_{s\alpha}]\|_{L_t^2 L_x^{\frac{8}{3}}} &\lesssim \|(s')^{\frac{1}{4}} A\|_{L_t^\infty L_x^8} \|(s')^{\frac{1}{2}} \psi_s\|_{L_t^2 L_x^8} \|(s')^{\frac{1}{4}} \Psi\|_{L_t^\infty L_x^8} \\ &\lesssim \|A\|_{L_t^\infty L_x^4}^{\frac{1}{2}} \|(s')^{\frac{1}{2}} \nabla A\|_{L_t^\infty L_x^4}^{\frac{1}{2}} \|(s')^{\frac{1}{2}} \psi_s\|_{L_t^2 L_x^8} \|(s')^{\frac{1}{4}} \Psi\|_{L_t^\infty L_x^8} \\ &\lesssim \epsilon^2 \|(s')^{\frac{1}{2}} \psi_s\|_{L_t^2 L_x^8} \|(s')^{\frac{1}{4}} \Psi\|_{L_t^\infty L_x^8} \end{aligned}$$

where the last line follows from Sobolev embedding, (3.89), and (3.91). Plugging all of the preceding estimates back into (6.17) we obtain

$$\begin{aligned} \|\nabla^\alpha A_\alpha\|_{L_t^\infty L_t^2 L_x^{\frac{8}{3}}} &\lesssim \int_0^\infty \|(s')^{\frac{1}{2}} \psi_s\|_{L_t^2 L_x^8} \|(s')^{\frac{1}{4}} \Psi\|_{L_t^\infty L_x^8}^2 \frac{ds'}{s'} \\ &\quad + \int_0^\infty \|(s')^{\frac{7}{8}} \nabla_{t,x} \psi_s\|_{L_t^\infty L_x^{\frac{16}{5}}} \|(s')^{\frac{1}{8}} \Psi\|_{L_t^2 L_x^{16}} \frac{ds'}{s'} \\ &\quad + \int_0^\infty \|(s')^{\frac{1}{2}} \nabla \psi_s\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|s' \nabla^2 \psi_s\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|(s')^{\frac{1}{4}} \nabla w\|_{L_t^2 L_x^2} \frac{ds'}{s'} \\ &\quad + \epsilon^2 \int_0^\infty \|(s')^{\frac{1}{2}} \psi_s\|_{L_t^2 L_x^8} \|(s')^{\frac{1}{4}} \Psi\|_{L_t^\infty L_x^8} \frac{ds'}{s'} \\ &\lesssim \|\psi_s\|_{\mathcal{S}(I)}^2 \end{aligned}$$

Here the estimate on the last line follows by several application of Cauchy-Schwarz along with (5.23), (5.25), (5.27), and (6.6). This concludes the proof.  $\square$

## 7. PROOFS OF PROPOSITION 3.1 AND THEOREM 4.1

The first goal of this section is to complete the proof of Proposition 3.1 by closing the bootstrap (3.3) $\implies$ (3.4). We then conclude with the proof of Theorem 4.1.

We begin by establishing the following a priori estimates, which together with Proposition 6.1 can be used to deduce Proposition 3.1.

**Proposition 7.1.** *Let  $\psi_s$  denote the heat tension field defined in (3.94) and let  $\Psi = \psi_\alpha dx^\alpha$  as above. Then*

$$\|\mathbf{D}\Psi\|_{s=0} \|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)} \lesssim \|\psi_s\|_{S(I)}. \quad (7.1)$$

where the implicit constant above is independent of  $C_0$  in (3.3).

Before beginning the proofs of Proposition 7.1 and Proposition 3.1, we first note that once these results have been established we can also prove uniform estimates on higher derivatives of  $\Psi$  assuming appropriate control on higher Sobolev norms of the initial data  $(u_0, u_1)$ . As in Corollary 6.2 the key point here is that  $\|\psi_s\|_{S(I)}$  is a controlling norm for all of the dynamic variables. We state the following corollary of Proposition 6.1, Corollary 6.2 and Proposition 7.1, which will be needed in the proof of Theorem 4.1. We omit the proof as it is nearly identical to the proof of Proposition 7.1 after commuting the covariant Laplacian,  $\mathbf{D}^b \mathbf{D}_b$ , with (7.11) and integrating by parts to control the full third order covariant derivative  $\mathbf{D}^3 \Psi$  in  $L_x^2$ .

**Corollary 7.2.** *Let  $(u_0, u_1)$  be initial data as in (1.6), (1.7) with*

$$\|(du_0, u_1)\|_{H^3 \times H^3(\mathbb{H}^4; T\mathcal{N})} =: B_0 < \infty \quad (7.2)$$

and let  $(u, \partial_t u)$  be a smooth wave map on a time interval  $I \subset \mathbb{R}$  with initial data  $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ . Let  $\psi_s$  be the associated heat tension field and  $\Psi = \psi_\alpha dx^\alpha$  be the representation of  $d_{t,x}u$  in the caloric gauge as defined as above. Then there exists  $\epsilon_4 > 0$  with the following property. Let  $\epsilon_3 > 0$  be as in Corollary 6.2 and assume that  $\|\psi_s\|_{S(I)} = \epsilon < \epsilon_3 < \epsilon_4$  is small enough so that the hypothesis of Corollary 6.2 are satisfied so that

$$\|\Delta \psi_s\|_{S(I)} \lesssim B_0 \quad (7.3)$$

holds. Then  $\epsilon_4 > 0$  can be chosen small enough so that

$$\|\mathbf{D}^3 \Psi\|_{s=0} \|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)} \lesssim B_0. \quad (7.4)$$

with an implicit constant that is independent of  $I$ .

We now turn to the proofs of Proposition 7.1, Proposition 3.1, and Theorem 4.1. First we assume Proposition 7.1 and use it to deduce Proposition 3.1.

*Proof of Proposition 3.1 assuming Proposition 7.1.* Recall that Proposition 3.1 follows by showing that there exist constants  $\epsilon_1 > 0$  and  $C_0 > 0$  so that (3.3) implies (3.4) for all  $\epsilon \leq \epsilon_1$ . Requiring first that  $\epsilon, \epsilon_1 > 0$  satisfy  $\epsilon \leq \epsilon_1 < \epsilon_2$ , where  $\epsilon_2 > 0$  is the constant in Proposition 6.1 we can combine Proposition 6.1 and Proposition 7.1 to obtain

$$\begin{aligned} \|\mathbf{D}\Psi\|_{s=0} \|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)} &\lesssim \|\psi_s\|_{S(I)} \lesssim \|s^{\frac{1}{2}} \nabla_{t,x} \psi_s|_{t=0}\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2(L_x^2)} \\ &\lesssim \|(du_0, u_1)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} = \epsilon \end{aligned} \quad (7.5)$$

for an implicit constant that is independent of the choice of  $C_0$  in (3.3). Therefore it suffices to show that

$$\|\nabla d_{t,x}u\|_{s=0} \|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)} \lesssim \|\mathbf{D}\Psi\|_{s=0} \|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)} \quad (7.6)$$

for an implicit constant independent of  $C_0$ . Finally to prove (7.6) we note that by using the schematic  $e(\mathbf{D}\Psi) = Dd_{t,x}u = \nabla d_{t,x}u + S(u)(du, d_{t,x}u)$ , we can show that for any time interval  $J \subset I$  we have

$$\|\nabla d_{t,x}u\|_{s=0} \|_{L_t^\infty L_x^2(J \times \mathbb{H}^4)} \lesssim \|\mathbf{D}\Psi\|_{s=0} \|_{L_t^\infty L_x^2(J \times \mathbb{H}^4)} + \|\nabla d_{t,x}u\|_{s=0} \|_{L_t^\infty L_x^2(J \times \mathbb{H}^4)}^2.$$

The estimate (7.6) now follows from the usual continuity argument. To conclude we choose  $C_0 > 0$  sufficiently large to account for the constants in (7.5) and (7.6) and then  $\epsilon_1 > 0$  sufficiently small relative to  $C_0$ . This completes the proof.  $\square$

Before turning to the proof of Proposition 7.1 we state two auxiliary lemmas which will be required for the proof.

**Lemma 7.3.** *Let  $\psi_s$  be as in the statement of Proposition 7.1. Then*

$$\begin{aligned} \|s^{\frac{1}{2}} \mathbf{D}_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} &\lesssim \|\psi_s\|_{S(I)}, \\ \|s \mathbf{D} \mathbf{D}_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} &\lesssim \|\psi_s\|_{S(I)}, \\ \|s^{\frac{3}{2}} \mathbf{D}^2 \mathbf{D}_{t,x} \psi_s\|_{L_{\frac{ds}{s}}^\infty \cap L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} &\lesssim \|\psi_s\|_{S(I)}. \end{aligned} \quad (7.7)$$

*Proof.* This is a consequence of Proposition 5.1, as well as bounds on  $A$  in (3.89) and Lemma 5.2. In order to replace  $\nabla$  in (5.1) by  $\mathbf{D}$  we note that for any smooth  $E_n$ -valued tensor  $\Phi$  we have  $\mathbf{D}\Phi = \nabla\Phi + A\Phi$ , and thus by Sobolev embedding,

$$\begin{aligned} \|\mathbf{D}_{t,x} \Phi\|_{L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} &\lesssim \|\nabla_{t,x} \Phi\|_{L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} + \|A\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4} \|\nabla \Phi\|_{L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \\ &\lesssim \|\nabla_{t,x} \Phi\|_{L_{\frac{ds}{s}}^2 L_t^\infty L_x^2}, \end{aligned} \quad (7.8)$$

Now apply (7.8) to  $\Phi = \psi_s, \mathbf{D}_{t,x} \psi_s$ , and  $\mathbf{D} \mathbf{D}_{t,x} \psi_s$ , followed by (3.89), (5.4) and Proposition 5.1.  $\square$

**Lemma 7.4.** *Let  $\psi_s$  and  $\Psi$  be as in the statement of Proposition 7.1. Then,*

$$\|s^{\frac{1}{2}} \mathbf{D}^2 \Psi\|_{L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \lesssim \|\psi_s\|_{S(I)}. \quad (7.9)$$

*Proof.* First note that for any  $a, b \in \{1, \dots, 4\}$  and  $\gamma \in \{0, 1, \dots, 4\}$  we have

$$\partial_s \mathbf{D}_a \mathbf{D}_b \psi_\gamma = F_{sa} \mathbf{D}_b \psi_\gamma + F_{sb} \mathbf{D}_a \psi_\gamma + \mathbf{D}_a F_{sb} \psi_\gamma + \mathbf{D}_a \mathbf{D}_b \mathbf{D}_\gamma \psi_s$$

from which it follows that

$$\begin{aligned} s^{\frac{1}{2}} \mathbf{D}_a \mathbf{D}_b \psi_\gamma(s) &= -s^{\frac{1}{2}} \int_s^\infty \mathbf{D}_a \mathbf{D}_b \mathbf{D}_\gamma \psi_s \, ds' - s^{\frac{1}{2}} \int_s^\infty F_{sa} \mathbf{D}_b \psi_\gamma \, ds' \\ &\quad - s^{\frac{1}{2}} \int_s^\infty F_{sb} \mathbf{D}_a \psi_\gamma \, ds' - s^{\frac{1}{2}} \int_s^\infty \mathbf{D}_a F_{sb} \psi_\gamma \, ds' \end{aligned}$$

Therefore, using (2.22) and (2.24) we have

$$\begin{aligned}
\|s^{\frac{1}{2}} \mathbf{D}^2 \Psi\|_{L_t^2 L_x^2}^2 &\leq \int_0^\infty \left[ \int_s^\infty \left( \frac{s}{s'} \right)^{\frac{1}{2}} \|(s')^{\frac{3}{2}} \mathbf{D}^2 \mathbf{D}_{t,x} \psi_s\|_{L_t^\infty L_x^2} \frac{ds'}{s'} \right]^2 \frac{ds}{s} \\
&\quad + \int_0^\infty \left[ \int_s^\infty \left( \frac{s}{s'} \right)^{\frac{1}{2}} \|(s')^{\frac{3}{2}} \mathbf{R}^{(0)}(\psi_s, \underline{\Psi}) \mathbf{D} \Psi\|_{L_t^\infty L_x^2} \frac{ds'}{s'} \right]^2 \frac{ds}{s} \\
&\quad + \int_0^\infty \left[ \int_s^\infty \left( \frac{s}{s'} \right)^{\frac{1}{2}} \|(s')^{\frac{3}{2}} \mathbf{R}^{(1)}(\underline{\Psi}; \psi_s, \underline{\Psi}) \Psi\|_{L_t^\infty L_x^2} \frac{ds'}{s'} \right]^2 \frac{ds}{s} \\
&\quad + \int_0^\infty \left[ \int_s^\infty \left( \frac{s}{s'} \right)^{\frac{1}{2}} \|(s')^{\frac{3}{2}} \mathbf{R}^{(0)}(\mathbf{D} \psi_s, \underline{\Psi}) \Psi\|_{L_t^\infty L_x^2} \frac{ds'}{s'} \right]^2 \frac{ds}{s} \\
&\quad + \int_0^\infty \left[ \int_s^\infty \left( \frac{s}{s'} \right)^{\frac{1}{2}} \|(s')^{\frac{3}{2}} \mathbf{R}^{(0)}(\psi_s, \mathbf{D} \underline{\Psi}) \Psi\|_{L_t^\infty L_x^2} \frac{ds'}{s'} \right]^2 \frac{ds}{s}
\end{aligned}$$

The lemma now follows from an application of Schur's test, (7.7), (5.27)–(5.28), and (5.2).  $\square$

*Proof of Proposition 7.1.* Recall that

$$\|\mathbf{D} \Psi \upharpoonright_{s=0}\|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)}^2 = \|\mathbf{D} \underline{\Psi} \upharpoonright_{s=0}\|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)}^2 + \|\mathbf{D} \psi_t \upharpoonright_{s=0}\|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)}^2$$

We prove

$$\|\mathbf{D} \underline{\Psi} \upharpoonright_{s=0}\|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)} \lesssim \|\psi_s\|_{S(I)} \quad (7.10)$$

as the proof of the same bound for  $\|\mathbf{D} \psi_t \upharpoonright_{s=0}\|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)}$  is nearly identical. Recall from Lemma 3.13 that for each  $a = 1, \dots, 4$ ,  $\psi_a$  satisfies the covariant heat equation

$$\partial_s \psi_a = \mathbf{D}_s \psi_a = \mathbf{D}^b \mathbf{D}_b \psi_a + 3\psi_a + F_a{}^b \psi_b \quad (7.11)$$

Using the shorthand  $\psi^a = \mathbf{h}^{ab} \psi_b$  and pairing the above with  $\mathbf{D}_c \mathbf{D}^c \psi^a$  gives

$$\langle \mathbf{D}_s \psi_a, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle = \langle \mathbf{D}^b \mathbf{D}_b \psi_a, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle + 3 \langle \psi_a, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle + \langle F_a{}^b \psi_b, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle$$

We expand the first term above as follows.

$$\begin{aligned}
\langle \mathbf{D}_s \psi_a, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle &= \nabla_c \langle \mathbf{D}_s \psi_a, \mathbf{D}^c \psi^a \rangle - \langle \mathbf{D}_c \mathbf{D}_s \psi_a, \mathbf{D}^c \psi^a \rangle \\
&= \nabla_c \langle \mathbf{D}_s \psi_a, \mathbf{D}^c \psi^a \rangle - \langle F_{cs} \psi_a, \mathbf{D}^c \psi^a \rangle - \frac{1}{2} \partial_s \langle \mathbf{D}_c \psi_a, \mathbf{D}^c \psi^a \rangle
\end{aligned}$$

Note furthermore that

$$\langle \psi_a, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle = \nabla_c \langle \psi_a, \mathbf{D}^c \psi^a \rangle - \langle \mathbf{D}_c \psi_a, \mathbf{D}^c \psi^a \rangle$$

From the previous three formulas, we have

$$\begin{aligned}
-\frac{1}{2} \partial_s \langle \mathbf{D}_c \psi_a, \mathbf{D}^c \psi^a \rangle &= -3 \langle \mathbf{D}_c \psi_a, \mathbf{D}^c \psi^a \rangle + \langle \mathbf{D}_b \mathbf{D}^b \psi_a, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle + \langle F_{cs} \psi_a, \mathbf{D}^c \psi^a \rangle \\
&\quad + \langle F_a{}^b \psi_b, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle - \nabla_c \langle \mathbf{D}_s \psi_a, \mathbf{D}^c \psi^a \rangle + 3 \nabla_c \langle \psi_a, \mathbf{D}^c \psi^a \rangle
\end{aligned}$$



Integrating over  $(s_1, s_2) \times \mathbb{H}^4$  gives,

$$\begin{aligned}
\frac{1}{2} \|\mathbf{D}\underline{\Psi}(s_1)\|_{L_x^2}^2 &= \frac{1}{2} \|\mathbf{D}\underline{\Psi}(s_2)\|_{L_x^2}^2 - 3 \int_{s_1}^{s_2} s \|\mathbf{D}\underline{\Psi}\|_{L_x^2}^2 \, \text{dvol}_{\mathbf{h}} \frac{ds}{s} \\
&\quad + \int_{s_1}^{s_2} \|s^{\frac{1}{2}} \mathbf{D}^b \mathbf{D}_b \underline{\Psi}\|_{L_x^2}^2 \frac{ds}{s} \\
&\quad + \int_{s_1}^{s_2} \int_{\mathbb{H}^4} s \langle F_{cs} \psi_a, \mathbf{D}^c \psi^a \rangle \, \text{dvol}_{\mathbf{h}} \frac{ds}{s} \\
&\quad + \int_{s_1}^{s_2} \int_{\mathbb{H}^4} s \langle F_a{}^b \psi_b, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle \, \text{dvol}_{\mathbf{h}} \frac{ds}{s}
\end{aligned} \tag{7.12}$$

First note that the second term on the right-hand side is non-positive, and thus can be ignored when proving an upper bound for the left-hand side. Next we claim that for each fixed  $t \in I$  we have

$$\lim_{s_2 \rightarrow \infty} \|\mathbf{D}\underline{\Psi}(s_2, t)\|_{L_x^2}^2 = 0 \tag{7.13}$$

This qualitative statement is a quick consequence of Poincaré's inequality, the identity  $e\mathbf{D}\underline{\Psi} = \nabla du + S(u)(du, du)$  and Proposition 3.5. Indeed, we have

$$\begin{aligned}
\|\mathbf{D}\underline{\Psi}(s, t)\|_{L_x^2} &\lesssim \|\nabla du(s, t)\|_{L_x^2} + \|S(u)(du, du)(s, t)\|_{L_x^2} \\
&\lesssim s^{-\frac{1}{2}} \|s^{\frac{1}{2}} \nabla^2 du(s, t)\|_{L_x^2} + s^{-1} \|s^{\frac{1}{2}} \nabla^2 du(s, t)\|_{L_x^2}^2
\end{aligned}$$

where on the last line we used the Sobolev inequality followed by the Poincaré inequality. By Proposition 3.5, the right-hand side vanishes as  $s \rightarrow \infty$  as desired.

Therefore, by letting  $s_2 \rightarrow \infty$  in (7.12), then taking the supremum over  $s_1$ , and then the supremum over  $t \in I$  yields the inequality

$$\begin{aligned}
\frac{1}{2} \|\mathbf{D}\underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^2 &\leq \int_0^\infty \|s^{\frac{1}{2}} \mathbf{D}^2 \underline{\Psi}(s)\|_{L_t^\infty L_x^2}^2 \frac{ds}{s} \\
&\quad + \sup_{t \in I} \int_0^\infty \int_{\mathbb{H}^4} |s \langle F_{cs} \psi_a, \mathbf{D}^c \psi^a \rangle| \, \text{dvol}_{\mathbf{h}} \frac{ds}{s} \\
&\quad + \sup_{t \in I} \int_0^\infty \int_{\mathbb{H}^4} |s \langle F_a{}^b \psi_b, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle| \, \text{dvol}_{\mathbf{h}} \frac{ds}{s}
\end{aligned} \tag{7.14}$$

From (7.14) we claim the estimate

$$\|\mathbf{D}\underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^2 \lesssim \|\psi_s\|_{S(I)}^2 + \epsilon \|\mathbf{D}\underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^2 \tag{7.15}$$

Note that (7.15) implies (7.10) since the second term on the right above can be absorbed into the left-hand side.

To prove (7.15) we estimate each of the terms on the right-hand side of (7.14). To begin the first, main term is handled by Lemma 7.4,

$$\int_0^\infty \|s^{\frac{1}{2}} \mathbf{D}^2 \underline{\Psi}(s)\|_{L_t^\infty L_x^2}^2 \frac{ds}{s} \lesssim \|\psi_s\|_{S(I)}^2$$

For the second term we note that

$$F_{cs} \psi_a = \mathbf{R}^{(0)}(\psi_c, \psi_s) \psi_a$$

and thus

$$|s \langle F_{cs} \psi_a, \mathbf{D}^c \psi^a \rangle| \lesssim |\underline{\Psi}|^2 |s^{\frac{1}{2}} \psi_s| |s^{\frac{1}{2}} \mathbf{D}\underline{\Psi}|$$

It follows that

$$\begin{aligned}
& \sup_{t \in I} \int_0^\infty \int_{\mathbb{H}^4} |s \langle F_{cs} \psi_a, \mathbf{D}^c \psi^a \rangle| \, \mathrm{dvol}_{\mathbf{h}} \frac{ds}{s} \\
& \lesssim \|\underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4}^2 \int_0^\infty \|s^{\frac{1}{2}} \psi_s\|_{L_t^\infty L_x^4} \|s^{\frac{1}{2}} \mathbf{D} \underline{\Psi}\|_{L_t^\infty L_x^4} \frac{ds}{s} \\
& \lesssim \epsilon^2 \|\psi_s\|_{S(I)}^2
\end{aligned} \tag{7.16}$$

where we have used Sobolev embedding and (3.87) to get the factor of  $\epsilon^2$  in the last line above and (5.28).

To bound the third term on the right-hand side of (7.14), we note that

$$F_a^b \psi_b = \mathbf{R}^{(0)}(\psi_a, \psi^b) \psi_b$$

Therefore,

$$|s \langle F_a^b \psi_b, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle| \lesssim |\underline{\Psi}|^3 |\mathbf{D}^2 \underline{\Psi}|$$

It follows that

$$\begin{aligned}
& \sup_{t \in I} \int_0^\infty \int_{\mathbb{H}^4} |s \langle F_a^b \psi_b, \mathbf{D}_c \mathbf{D}^c \psi^a \rangle| \, \mathrm{dvol}_{\mathbf{h}} \frac{ds}{s} \\
& \lesssim \|\underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^4}^2 \sup_{t \in I} \int_0^\infty \|s^{\frac{1}{2}} \underline{\Psi}(s, t)\|_{L_x^\infty} \|s^{\frac{1}{2}} \mathbf{D}^2 \underline{\Psi}\|_{L_t^\infty L_x^2} \frac{ds}{s} \\
& \lesssim \|\nabla \underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^2 \|s^{\frac{1}{2}} \underline{\Psi}\|_{L_t^\infty L_{\frac{ds}{s}}^2 L_x^\infty} \|s^{\frac{1}{2}} \mathbf{D}^2 \underline{\Psi}\|_{L_{\frac{ds}{s}}^2 L_t^\infty L_x^2}
\end{aligned} \tag{7.17}$$

By (3.89), we have

$$\|\nabla \underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2} \lesssim \|\mathbf{D} \underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}$$

Using the previous inequality together with (5.2), Lemma 7.3, and Proposition 6.1, (7.17) can be estimated by

$$\begin{aligned}
& \|\nabla \underline{\Psi}\|_{L_{\frac{ds}{s}}^\infty L_t^\infty L_x^2}^2 \|s^{\frac{1}{2}} \underline{\Psi}\|_{L_t^\infty L_{\frac{ds}{s}}^2 L_x^\infty} \|s^{\frac{1}{2}} \mathbf{D}^2 \underline{\Psi}\|_{L_{\frac{ds}{s}}^2 L_t^\infty L_x^2} \lesssim \epsilon \|\psi_s\|_{S(I)} \|\mathbf{D} \underline{\Psi}\|_{L_t^\infty L_{\frac{ds}{s}}^\infty L_x^2}^2 \\
& \lesssim \epsilon^2 \|\mathbf{D} \underline{\Psi}\|_{L_t^\infty L_{\frac{ds}{s}}^\infty L_x^2}^2
\end{aligned}$$

which can be absorbed into the left-hand side of (7.14) as long as  $\epsilon > 0$  is small enough. This completes the proof of (7.10) and the proof of Proposition 7.1.  $\square$

Now that we have established the a priori estimates in Proposition 3.1 we can complete the proof of Theorem 4.1. The key point is that the conclusion of Proposition 6.1 now holds unconditionally.

*Proof of Theorem 4.1 and of Theorem 1.1.* We begin by proving the global existence and regularity statement in Theorem 4.1. This is a standard argument now that we have established the a priori estimates in Proposition 3.1 and Proposition 6.1 and so we give only a brief sketch.

Let  $\epsilon_0 = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ , where the epsilons on the right-hand side are as in Proposition 3.1, Proposition 6.1, Corollary 6.2 and Corollary 7.2, respectively. Let  $(u_0, u_1)$  be any smooth initial data set satisfying (1.6), (1.7) and with

$$\|(du_0, u_1)\|_{H^1 \times H^1(\mathbb{H}^4; T\mathcal{N})} = \epsilon < \epsilon_0 \tag{7.18}$$

Our assumption that the data is smooth and constant outside a compact subset of  $\mathbb{H}^4$  means that we can further assume that, say,

$$\|(du_0, u_1)\|_{H^3 \times H^3(\mathbb{H}^4; T\mathcal{N})} = B_0 < \infty. \quad (7.19)$$

Using the usual argument based on energy estimates one can find a positive time  $T > 0$  and a unique solution  $(u, \partial_t u) \in C([0, T]; H^4 \times H^3(\mathbb{H}^4; T\mathcal{N}))$ . Moreover, the time of existence  $T > 0$  depends only  $B_0$  from (7.19). One then arrives at a finite time blow-up criterion: if the solution  $(u, \partial_t u)$  cannot be smoothly extended past a finite time  $T^* < \infty$ , then

$$\limsup_{t \nearrow T^*} \|(du, \partial_t u)(t)\|_{H^3 \times H^3(\mathbb{H}^4; T\mathcal{N})} = +\infty. \quad (7.20)$$

We will show that (7.20) is impossible for  $T_* < \infty$  under the hypothesis of Theorem 4.1. Assume for contradiction that  $(u, \partial_t u)$  with data as in (7.18) cannot be extended as a smooth solution beyond a time  $0 < T_* < \infty$  and denote by  $I$  the interval  $I = [0, T_*)$ . Since  $\epsilon_0 > 0$  in (7.18) is small enough so that Proposition 3.1 holds, we know by Proposition 6.1 that

$$\|\psi_s\|_{S(I)} \lesssim \epsilon \lesssim \epsilon_0. \quad (7.21)$$

Note that Proposition 3.1 along with (7.21) imply that all of the additional estimates from Sections 5, 6, and 7 hold true. Moreover, since  $\epsilon_0$  is small enough so that the hypothesis of Corollary 6.2 and Corollary 7.2 are satisfied, we obtain the uniform bounds

$$\|\Delta\psi_s\|_{S(I)} \lesssim B_0, \quad \|\mathbf{D}^3\Psi\|_{s=0} \|_{L_t^\infty L_x^2(I \times \mathbb{H}^4)} \lesssim B_0. \quad (7.22)$$

But then we can argue as in the proof of Proposition 3.1, using the smallness

$$\|(du, \partial_t u)\|_{s=0} \|_{L_t^\infty(I; H^1 \times H^1(\mathbb{H}^4))} \lesssim \epsilon_0$$

to conclude that in fact

$$\sup_{t \in I} \|(du, \partial_t u)(t)\|_{H^3 \times H^3(\mathbb{H}^4; T\mathcal{N})} \lesssim B_0, \quad (7.23)$$

which contradicts (7.20). The same argument applied to negative times shows that any smooth solution  $(u, \partial_t u)$  with sufficiently small data as in the statement of Theorem 4.1 is defined and globally regular on  $I = \mathbb{R}$ .

Since Proposition 3.1 holds now with  $I = \mathbb{R}$ , the statements (4.5) and (4.7) follow from Proposition 6.1 and Corollary 6.2. Moreover, the scattering statement (4.6) is a standard consequence of the argument used to prove Proposition 6.1.

It remains to establish (4.8). Let  $u(s, t, x)$  denote the harmonic map heat flow resolution of the wave map  $u(t, x)$  and we write  $u(0, t, x) = u(t, x)$ . By Lemma 3.2 we know that each fixed  $(t, x) \in \mathbb{R} \times \mathbb{H}^4$  we have

$$u(s, t, x) \rightarrow u_\infty \quad \text{as } s \rightarrow \infty.$$

It follows that for each fixed  $(t, x) \in \mathbb{R} \times \mathbb{H}^4$

$$|u(t, x) - u_\infty| = |u(0, t, x) - u_\infty| = \left| \int_0^\infty \partial_s u(s, t, x) ds \right|.$$

Recalling that  $\partial_s u = e\psi_s$ , we see that

$$\lim_{t \rightarrow \infty} \|u(t) - u_\infty\|_{L_x^\infty} \leq \lim_{t \rightarrow \infty} \int_0^\infty \|\partial_s u(s, t)\|_{L_x^\infty} ds = \lim_{t \rightarrow \infty} \int_0^\infty \|\psi_s(s, t)\|_{L_x^\infty} ds.$$

From the above we observe that the proof of (4.8) is now a consequence of the following two statements and the dominated convergence theorem.

- (a) There exists an integrable function  $f(s) \geq 0$ ,  $\int_0^\infty f(s) ds < \infty$  so that

$$\|\partial_s u(s, t)\|_{L_x^\infty} = \|\psi_s(s, t)\|_{L_x^\infty} \leq f(s), \quad (7.24)$$

uniformly in  $t \in \mathbb{R}$ .

- (b) For each fixed  $s \in (0, \infty)$  we have

$$\|\psi_s(s, t)\|_{L_x^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (7.25)$$

To prove (a) we define  $f(s)$  simply by  $f(s) := \|\psi_s(s)\|_{L_{t,x}^\infty}$ . First, we estimate  $f(s)$  on the interval  $s \in (0, 1]$ . The key ingredient in this region is the higher regularity estimate (4.7), which breaks the scaling in  $s$ . Indeed, by Gagliardo-Nirenberg we have

$$\begin{aligned} \|\psi_s(s)\|_{L_{t,x}^\infty} &\lesssim \|\psi_s(s)\|_{L_t^\infty L_x^8}^{\frac{1}{2}} \|(-\Delta)^{\frac{1}{2}} \psi_s(s)\|_{L_t^\infty L_x^8}^{\frac{1}{2}} \\ &\lesssim s^{-\frac{3}{4}} \|s^{\frac{1}{2}} \nabla \psi_s(s)\|_{L_t^\infty L_x^2}^{\frac{1}{4}} \|s \nabla^2 \psi_s(s)\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|s^{\frac{1}{2}} \nabla \Delta \psi_s(s)\|_{L_t^\infty L_x^2}^{\frac{1}{4}} \\ &\lesssim s^{-\frac{3}{4}} \|\psi_s\|_{\mathcal{S}(\mathbb{R})}^{\frac{3}{4}} \|\Delta \psi_s\|_{\mathcal{S}(\mathbb{R})}^{\frac{1}{4}} \lesssim s^{-\frac{3}{4}}, \end{aligned} \quad (7.26)$$

where the last two lines follows from the definition of the norm  $\mathcal{S}(\mathbb{R})$ , (4.5), (4.7), and (5.1). Therefore,

$$\int_0^1 \|\psi_s(s)\|_{L_{t,x}^\infty} ds \lesssim \int_0^1 s^{-\frac{3}{4}} ds \lesssim 1.$$

For the interval  $s \in [1, \infty)$  we note that (3.53) provides, say, the bound

$$\|\psi_s(s)\|_{L_{t,x}^\infty} \lesssim s^{-2} \quad \text{as } s \rightarrow \infty$$

so that

$$\int_1^\infty \|\psi_s(s)\|_{L_{t,x}^\infty} ds \lesssim 1,$$

which completes the proof of (a).

Next, we prove (b). Fix  $s_0 > 0$ . By an approximation argument, we may assume that the scattering profile  $\phi_+(s_0)$  in (4.6) is smooth with compactly supported initial data  $\vec{\phi}_+(s_0, 0)$ . For such a free wave, i.e., for  $\phi_+(s_0, t)$ , we can deduce from the Gagliardo-Nirenberg inequality and the global dispersive estimate (4.14) for the free propagator that

$$\|\phi_+(s_0, t)\|_{L_x^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (7.27)$$

Next, again using Gagliardo-Nirenberg, we have

$$\begin{aligned} &\|\psi_s(s_0, t) - \phi_+(s_0, t)\|_{L_x^\infty} \\ &\lesssim \|\nabla(\psi_s(s_0, t) - \phi_+(s_0, t))\|_{L_x^2}^{\frac{1}{2}} \left( \|\nabla^3 \psi_s(s_0)\|_{L_t^\infty L_x^2} + \|\nabla^3 \phi_+(s_0)\|_{L_t^\infty L_x^2} \right)^{\frac{1}{2}} \\ &\lesssim C_{s_0, \phi_+} \|\nabla(\psi_s(s_0, t) - \phi_+(s_0, t))\|_{L_x^2}^{\frac{1}{2}} \longrightarrow 0 \quad \text{as } t \rightarrow \infty, \end{aligned}$$

where we have used (5.1) to control the first term in parenthesis for fixed  $s_0 > 0$ , and then (4.6) to obtain the convergence in the last line. This completes the proof of Theorem 4.1 and thus of Theorem 1.1 as well.  $\square$

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